Flat and étale morphisms

Maxim Mornev

All rings are commutative.

Contents

1 Flat morphisms

1.1 Preliminaries on tensor product

Let A be a ring, M and A-module. For all A-modules N_1, N_2 we have a natural isomorphism

$$\mathrm{Hom}_A(N_1 \otimes_A M, N_2) \cong \mathrm{Hom}_A(N_1, \mathrm{Hom}_A(M, N_2)).$$

In other words $\otimes_A M$ is left adjoint to $\mathrm{Hom}_A(M, -)$. Hence $\otimes_A M$ is right exact and commutes with colimits.

Left derived functors $L^i(\otimes_A M)(-)$ are denoted $\mathrm{Tor}_i^A(-, M)$. A morphism of modules $M \to M'$ induces natural morphisms $\mathrm{Tor}_i^A(-, M) \to \mathrm{Tor}_i^A(-, M')$, so Tor_i is a bifunctor. The most important property of Tor is its commutativity:

Theorem 1.1.1. *Let A be a ring, and let M, N be A-modules. For every $i \geqslant 0$ there exists a natural isomorphism $\mathrm{Tor}_i^A(N, M) \to \mathrm{Tor}_i^A(M, N)$.*[1]

We will not need the full force of this theorem and so omit its proof.

Proposition 1.1.2. *Let A be a ring, $I \subset A$ an ideal, and M an A-module.* $\mathrm{Tor}_1^A(A/I, M) = \ker(I \otimes_A M \to M)$.

Proof. The short exact sequence $0 \to I \to A \to A/I \to 0$ induces an exact sequence $0 = \mathrm{Tor}_1^A(A, M) \to \mathrm{Tor}_1^A(A/I, M) \to I \otimes_A M \to M$. □

Corollary 1.1.3. *Let $a \in A$ be a nonzero element.* $\mathrm{Tor}_1^A(A/(a), M)$ *is the a-torsion of M.*

Let A, B be rings, N_1 an A-module, N_2 an A, B-bimodule, and N_3 a B-module. There is an isomorphism of A, B-bimodules

$$(N_1 \otimes_A N_2) \otimes_B N_3 \to N_1 \otimes_A (N_2 \otimes_B N_3),$$

which is natural in N_1, N_2, N_3.

Also recall that if A is a ring and $S \subset A$ a multiplicative system, then the functor $\otimes_A A_S$ is isomorphic to the functor of localization at S.

1.2 Flat modules

Definition 1.2.1. Let A be a ring. A module M over A is called flat if $\otimes_A M$ is exact.

Proposition 1.2.2. *Let $A \to B$ be a morphism of rings, and M a B-module. If M is flat over B and B is flat over A then M is flat over A.*

Proof. The functor $- \otimes_A M$ is isomorphic to the composition $(- \otimes_A B) \otimes_B M$ of exact functors. □

Proposition 1.2.3. *Let $A \to B$ be a morphism of rings. If M is a flat A-module, then $B \otimes_A M$ is a flat B-module.*

Proof. The functor $- \otimes_B (B \otimes_A M)$ is isomorphic to the functor $- \otimes_A M$, which is exact. □

[1] See [3], chapter 2, section 2.7

Proposition 1.2.4. *Let $\varphi\colon A \to B$ be a morphism of rings, and M a B-module. M is flat over A if and only if for every $\mathfrak{q} \in \operatorname{Specmax} B$ the module $M_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$, where $\mathfrak{p} = \varphi^{-1}\mathfrak{q}$.*

Proof. Notice that $\otimes_A M$ sends A-modules to B-modules, with the structure of B-module inherited from M. Let $\mathfrak{q} \in \operatorname{Specmax} B$, and $\mathfrak{p} = \varphi^{-1}\mathfrak{q}$. We have an isomorphism of functors from the category of A-modules to the category of $B_{\mathfrak{q}}$-modules:

$$(- \otimes_A M)_{\mathfrak{q}} = (- \otimes_A M) \otimes_B B_{\mathfrak{q}} = - \otimes_A (M \otimes_B B_{\mathfrak{q}}) = \\ - \otimes_A M_{\mathfrak{q}} = - \otimes_A (A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{q}}) = (-)_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{q}}.$$

Localization is exact. Hence, if $\otimes_A M$ is flat then $\otimes_{A_{\mathfrak{p}}} M_{\mathfrak{q}}$ is exact. Conversely, if $\otimes_{A_{\mathfrak{p}}} M_{\mathfrak{q}}$ is exact for every $\mathfrak{q} \in \operatorname{Specmax} B, \mathfrak{p} = \varphi^{-1}\mathfrak{q}$, then tensoring a short exact sequence $N_1 \to N_2 \to N_3$ with M we obtain a sequence of B-modules which is exact at every maximal ideal $\mathfrak{q}$. Therefore it is exact. □

Proposition 1.2.5. *Let A be a ring. An A-module is flat if and only if $I \otimes_A M \to M$ is injective (equivalently, $\operatorname{Tor}_1^A(A/I, M) = 0$) for every finitely generated ideal $I \subset A$.*

Proof. The "only if" part is trivial. We want to show that for arbitrary inclusion of A-modules $N' \subset N$ the induced morphism $N' \otimes_A M \to N \otimes_A M$ is injective.

We first show that $I \otimes_A M \to M$ is injective for every ideal I. Let $x \in I \otimes_A M$ be an element which vanishes in M. The element x is a finite linear combination of elementary tensors $y \otimes m$ where $y \in I, m \in M$. Thus there exists a finitely generated ideal $I' \subset I$ and $x' \in I' \otimes_A M$ such that the image of x' in $I \otimes_A M$ is equal to x. The map $I' \otimes_A M \to M$ is injective, so $x' = 0$ and hence $x = 0$, i.e. $I \otimes_A M \to M$ is injective. As a corollary, $\operatorname{Tor}_1^A(N, M) = 0$ if N is a cyclic module, that is, $N = A/I$ for some ideal $I \subset A$.

Let N be an arbitrary module and N' its submodule. Consider an index set J whose elements are finite subsets of $N \backslash N'$. For $j \in J$ let N_j be the submodule of N generated by N' and j. Jf $j \subset j'$ then there is a natural injection $N_j \to N_{j'}$. The inclusion order on J makes it a directed poset. Clearly, $\operatorname{colim}_{j \in J} N_j = N$.

Let $j \subset j'$ be an inclusion. Assume that $j' \backslash j$ consists of a single element. In this case $N_{j'}/N_j$ is a cyclic module. The short exact sequence $0 \to N_j \to N_{j'} \to N_{j'}/N_j \to 0$ induces an exact sequence $\operatorname{Tor}_1^A(N_{j'}/N_j, M) \to N_j \otimes_A M \to N_{j'} \otimes_A M$. Since $N_{j'}/N_j$ is cyclic, $\operatorname{Tor}_1^A(N_{j'}/N_j, M)$ vanishes, and so $N_j \otimes_A M \to N_{j'} \otimes_A M$ is injective.

A general inclusion $j \subset j'$ can be factored into a sequence of inclusions such that at each step only one new element appears. Hence $N_j \otimes_A M \to N_{j'} \otimes_A M$ is injective, which implies that the morphism $N' \otimes_A M \to \operatorname{colim}_{j \in J} N_j \otimes_A M$ is injective too. It remains to recall that $\otimes_A M$ commutes with colimits. □

Corollary 1.2.6. *Let A be a PID. An A-module M is flat if and only if it is torsion-free.*

Proposition 1.2.7. *Let A be a ring, let $0 \to M' \to M'' \to M \to 0$ be a short exact sequence of A-modules, and let N be an A-module. If M is flat then $M' \otimes_A N \to M'' \otimes_A N$ is injective.*

Proof. One can either refer to commutativity of Tor or do a direct proof as follows. Let $0 \to K \to F \to N \to 0$ be a short exact sequences with F a free module. Consider a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & M' \otimes_A K & \longrightarrow & M'' \otimes_A K & \longrightarrow & M \otimes_A K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' \otimes_A F & \longrightarrow & M'' \otimes_A F & \longrightarrow & M \otimes_A F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & M' \otimes_A N & \longrightarrow & M'' \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

A simple diagram chase finishes the proof. □

Theorem 1.2.8. *Let A be a local noetherian ring, and M an A-module of finite type. If M is flat then it is free.*

Proof. Let k be the residue field of A. Take a k-basis of $M \otimes_A k$. Lifting it to M we obtain a morphism from a free A-module F of finite type to M. By Nakayama lemma this morphism is surjective. Let K be its kernel. Tensoring the short exact sequence $0 \to K \to F \to M \to 0$ by k we obtain exact sequence $K \otimes_A k \to F \otimes_A k \to M \otimes_A k \to 0$. The morphism $K \otimes_A k \to F \otimes_A k$ is injective by proposition 1.2.7. The morphism $F \otimes_A k \to M \otimes_A k$ is an isomorphism by construction. Hence $K \otimes_A k$ is zero. On the other hand, K is of finite type since A is noetherian. So, Nakayama lemma shows that $K = 0$. □

1.3 Artin-Rees lemma and Krull intersection theorem

Let A be a ring, $I \subset A$ an ideal.

Definition 1.3.1. Let M be an A-module. An I-filtration on M is a descending chain of submodules $F_iM \subset M$, $i \in \mathbf{Z}_{\geqslant 0}$, such that $F_0M = M$ and $IF_iM \subset F_{i+1}M$ for every i.

Definition 1.3.2. Let M be an A-module. An I-filtration $F_i M$ is called stable if $IF_i M = F_{i+1}M$ for sufficiently large i.

Proposition 1.3.3. *Let A be a ring, $I \subset A$ an ideal, and let N, M be A-modules. If $F_i N$ is a stable I-filtration of N then the filtration of $N \otimes_A M$ by images of $F_i N \otimes_A M$ is stable.*

Proof. Omitted. □

Proposition 1.3.4. *Let $A \to B$ be a morphism of rings, $I \subset A$ an ideal, M a B-module, and $F_i M$ a stable I-filtration of M as an A-module. If each $F_i M$ is a B-submodule, then $F_i M$ is a stable IB-filtration of M as a B-module.*

Proof. Omitted. □

Let M be an A-module endowed with an I-filtration $F_i M$. Consider a graded ring $B_I A = \bigoplus_{i=0}^{\infty} I^i$ and a $B_I A$-module $B_F M = \bigoplus_{i=0}^{\infty} F_i M$.

Proposition 1.3.5. *Let A be a noetherian ring, $I \subset A$ an ideal, M an A-module with an I-filtration $F_i M$. The filtration is stable if and only if $B_F M$ is of finite type over $B_I A$.*

Lemma 1.3.6 (Artin-Rees lemma)**.** *Let A be a noetherian ring, $I \subset A$ an ideal, M an A-module with a stable I-filtration $F_i M$, and $N \subset M$ a submodule. The filtration $F_i N = N \cap F_i M$ is stable.*

Proof. The ring $B_I A$ is noetherian since it is a quotient of the polynomial ring $A[x_1, \dots, x_n]$ for some n. The module $B_F N$ is a submodule of $B_F M$, and thus is of finite type. Now the claim follows from the previous proposition. □

Theorem 1.3.7 (Krull intersection theorem)**.** *Let A be a noetherian local ring, $I \subset A$ an ideal and M a module of finite type. If $F_i M$ is a stable I-filtration of M, then $\bigcap_{i=0}^{\infty} F_i M = 0$.*

Proof. Consider the submodule $N = \bigcap_{i=0}^{\infty} F_i M$. By construction $N \cap F_i M = N$ for every i, and so by Artin-Rees lemma $N = IN$. Hence $N = \mathfrak{m}N$. Since N is of finite type, Nakayama lemma implies that $N = 0$. □

1.4 Modules of finite length

Let A be a ring, M a module. A strict chain of submodules of length n is an increasing sequence of submodules of M:

$$M_0 \subset M_1 \subset \ldots \subset M_n,$$

such that $M_0 = 0, M_n = M$, and each inclusion $M_i \subset M_{i+1}$ is nontrivial.

We define $l_A(M)$, the length of M, as the supremum of lengths of strict chains.

Definition 1.4.1. M is called a module of finite length if $l_A(M)$ is finite (i.e. if the supremum exists).

Proposition 1.4.2. *$l_A(M) = 1$ if and only if $M = A/\mathfrak{m}$ for some $\mathfrak{m} \in \operatorname{Specmax} A$.*

Proof. Excercise. ☐

Proposition 1.4.3. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. If M is of finite length or M' and M'' are of finite length then all three modules are of finite length and $l_A(M) = l_A(M') + l_A(M'')$.*

Proof. Excercise. ☐

Proposition 1.4.4. *Let A be a ring, $\mathfrak{m} \subset A$ a maximal ideal of finite type, and M an A-module of finite type. If $\mathfrak{m}^n M = 0$ for some $n > 0$, then M is of finite length.*

Proof. Let $n > 0$ be an integer. Suppose that $A/\mathfrak{m}^n$ is of finite length. If a module M of finite type is annihilated by $\mathfrak{m}^n$ then it is an $A/\mathfrak{m}^n$-module, and so is a quotient of a finite direct sum of $A/\mathfrak{m}^n$'s. Hence M is of finite length.

We next prove that $A/\mathfrak{m}^n$ is of finite length using induction over n. The case $n = 1$ was already established. Consider a short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^n \to A/\mathfrak{m}^n \to A/\mathfrak{m} \to 0.$$

The module $\mathfrak{m}/\mathfrak{m}^n$ is of finite type since $\mathfrak{m}$ is, and is annihilated by $\mathfrak{m}^{n-1}$, whence of finite length. But then $A/\mathfrak{m}^n$ is also of finite length. ☐

Proposition 1.4.5. *Let A be a ring, M an A-module. If $\operatorname{Tor}_1^A(A/\mathfrak{m}, M) = 0$ for every $\mathfrak{m} \in \operatorname{Specmax} A$, then $\operatorname{Tor}_1^A(N, M) = 0$ for every module N of finite length.*

Proof. We will do it by induction on $l_A(N)$. If $l_A(N) = 1$ then N is of the form $A/\mathfrak{m}$, and so $\operatorname{Tor}_1^A(N, M) = 0$ by assumption. Otherwise there exists a proper nontrivial submodule $N' \subset N$. Consider an exact sequence $\operatorname{Tor}_1^A(N', M) \to \operatorname{Tor}_1^A(N, M) \to \operatorname{Tor}_1^A(N/N', M)$ induced by short exact sequence $0 \to N' \to N \to N/N' \to 0$. Since $l_A(N') < l_A(N)$ and $l_A(N/N') < l_A(N)$, we see that $\operatorname{Tor}_1^A(N', M) = \operatorname{Tor}_1^A(N/N', M) = 0$, so $\operatorname{Tor}_1^A(N, M) = 0$. ☐

1.5 Criteria of flatness

Theorem 1.5.1 (Critère local de platitude)**.** *Let $A \to B$ be a local morphism of noetherian local rings, k the residue field of A, and M a B-module of finite type. If $\operatorname{Tor}_1^A(k, M) = 0$ then M is flat over A.*

Proof. We want to show that for every ideal $I \subset A$ the module $\mathrm{Tor}_1^A(A/I, M)$ vanishes. Notice that if A/I is of finite length, then $\mathrm{Tor}_1^A(A/I, M) = 0$ by proposition 1.4.5.

Let $\mathfrak{m} \subset A$ be the maximal ideal, and $I \subset A$ an arbitrary ideal. Let $n > 0$ be an integer. Consider a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \cap \mathfrak{m}^n & \longrightarrow & I & \longrightarrow & I/(I \cap \mathfrak{m}^n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{m}^n & \longrightarrow & A & \longrightarrow & A/\mathfrak{m}^n & \longrightarrow & 0. \end{array}$$

Tensoring it with M over A we obtain a diagram

$$\begin{array}{ccccc} (I \cap \mathfrak{m}^n) \otimes_A M & \longrightarrow & I \otimes_A M & \longrightarrow & (I/(I \cap \mathfrak{m}^n)) \otimes_A M \\ \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta_n} \\ \mathfrak{m}^n \otimes_A M & \longrightarrow & M & \longrightarrow & (A/\mathfrak{m}^n) \otimes_A M. \end{array}$$

with right exact rows. The cokernel of the map $I/(I \cap \mathfrak{m}^n) \to A/\mathfrak{m}^n$ is $A/(I + \mathfrak{m}^n)$. It has finite length by proposition 1.4.4. Thus $\mathrm{Tor}_1^A(A/(I + \mathfrak{m}^n), M) = 0$ and the morphism β_n is injective. As a consequence, $\ker(\alpha)$ is contained in the image of $(I \cap \mathfrak{m}^n) \otimes_A M$.

The filtration $\mathfrak{m}^n$ on A is $\mathfrak{m}$-stable. Hence by Artin-Rees lemma the filtration $I \cap \mathfrak{m}^n$ on I is $\mathfrak{m}$-stable, and so the filtration on $I \otimes_A M$ by images of $(I \cap \mathfrak{m}^n) \otimes_A M$ is $\mathfrak{m}$-stable (notice that $I \otimes_A M$ is not necessarily an A-module of finte type!).

The module $I \otimes_A M$ has a structure of B-module via M, and the images of $(I \cap \mathfrak{m}^n) \otimes_A M$ in this module are B-submodules. Let $J = \mathfrak{m}B \subset B$. This ideal is proper since $A \to B$ is a local morphism. The filtration on $I \otimes_A M$ as a B-module is J-stable. Now, Krull intersection theorem tells us that $\ker(\alpha) = 0$ as a submodule of zero module. □

Lemma 1.5.2. *Let $A \to B$ be a local morphism of local noetherian rings, $I \subset A$ an ideal, and M a B-module of finite type. If $\mathrm{Tor}_1^A(A/I, M) = 0$ and M/IM is a flat A/I-module, then M is a flat A-module.*

Proof. Let k be the residue field of A. A short exact sequence

$$0 \to K \to A/I \to k \to 0.$$

yields an exact sequence

$$\mathrm{Tor}_1^A(A/I, M) \to \mathrm{Tor}_1^A(k, M) \to K \otimes_A M \to A/I \otimes_A M$$

By assumptions $\mathrm{Tor}_1^A(A/I, M) = 0$. The modules K and A/I are A/I-modules, and the functor $\otimes_A M$ restricted to such modules is isomorphic to $\otimes_{A/I} M/IM$. The latter functor is exact, and so the arrow $K \otimes_A M \to A/I \otimes_A M$ is injective. Hence $\mathrm{Tor}_1^A(k, M) = 0$, and the local criterion of flatness finishes the proof. □

Proposition 1.5.3. *Let A be a ring, M a flat A-module. If $M/\mathfrak{m}M \neq 0$ for every $\mathfrak{m} \in \operatorname{Specmax} A$, then $N \otimes_A M = 0$ implies $N = 0$.*

Proof. If $\mathfrak{m} \in \operatorname{Specmax} A$, then

$$(N \otimes_A M) \otimes_A k(\mathfrak{m}) = N/\mathfrak{m}N \otimes_{k(\mathfrak{m})} M/\mathfrak{m}M.$$

Since $N \otimes_A M = 0$, we see that $N/\mathfrak{m}N = 0$ for every $\mathfrak{m} \in \operatorname{Specmax} A$. If N is of finite type, then by Nakayama $N_\mathfrak{m} = 0$ for every $\mathfrak{m} \in \operatorname{Specmax} A$, so $N = 0$. If N is not of finite type, then we take an element $x \in N$ and consider a submodule N' generated by x. The morphism $N' \to N$ is injective, so $N' \otimes_A M \to N \otimes_A M$ is injective, and as a consequence $N' = 0$, i.e. $x = 0$. Hence, $N = 0$. □

Theorem 1.5.4 (Critère de platitude par fibres, cas noethérien)**.** *Let $A \to B \to C$ be local morphisms of local noetherian rings, and M a C-module of finite type. Let k be the residue field of A. If M is nonzero, flat over A, and $M \otimes_A k$ is flat over $B \otimes_A k$, then B is flat over A and M is flat over B.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A, and $I = \mathfrak{m}B$. The natural map $\mathfrak{m} \otimes_A B \to I$ is surjective, and $(\mathfrak{m} \otimes_A B) \otimes_B C = \mathfrak{m} \otimes_A C$, so $\mathfrak{m} \otimes_A C \to I \otimes_B C$ is surjective. As a consequence, $\mathfrak{m} \otimes_A M \to I \otimes_B M$ is surjective.

The composition $\mathfrak{m} \otimes_A M \to I \otimes_B M \to M$ is injective, since M is flat over A. Hence $\mathfrak{m} \otimes_A M \to I \otimes_B M$ is an isomorphism, and $I \otimes_B M \to M$ is injective. In particular, $\operatorname{Tor}_1^A(B/I, M) = 0$, so M is flat over B by lemma 1.5.2.

Consider an exact sequence $0 \to \mathfrak{m} \to A \to k \to 0$. Tensoring with B over A gives us an exact sequence $0 \to \operatorname{Tor}_1^A(k, B) \to \mathfrak{m} \otimes_A B \to I \to 0$. Tensoring the latter sequence with M over B yields a sequence $0 \to \operatorname{Tor}_1^A(k, B) \otimes_B M \to \mathfrak{m} \otimes_A M \to I \otimes_B M \to 0$. The last map is an isomorphism, so $\operatorname{Tor}_1^A(k, B) \otimes_B M = 0$.

If $\mathfrak{m}_B \subset B$ and $\mathfrak{m}_C \subset C$ are maximal ideals, then $M/\mathfrak{m}_C M$ is nonzero by Nakayama, so $M/\mathfrak{m}_B M$ is nonzero. Hence, proposition 1.5.3 applies and shows that $\operatorname{Tor}_1^A(k, B) = 0$. It remains to apply theorem 1.5.1. □

1.6 Flatness in the context of schemes

Definition 1.6.1. Let $f\colon X \to Y$ be a morphism of schemes, and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is flat over Y at $x \in X$ if the stalk $\mathcal{F}_x$ is a flat module over $\mathcal{O}_{Y,f(x)}$. We say that f is flat at $x \in X$ if $\mathcal{O}_X$ is flat over Y at x. We say that $\mathcal{F}$ is flat over Y if it is flat over Y at all points. We say that f is flat if $\mathcal{O}_X$ is flat over Y.

Proposition 1.6.2. *Flat morphisms have following properties:*

(1) If X and Y are affine schemes and $\mathcal{F}$ is quasi-coherent, then $\mathcal{F}$ is flat over Y if and only if $\Gamma(X, \mathcal{F})$ is a flat module over $\Gamma(Y, \mathcal{O}_Y)$.

(2) Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms, and $\mathcal{F}$ a quasi-coherent sheaf. If $\mathcal{F}$ is flat over Y and g is flat, then $\mathcal{F}$ is flat over Z. In particular, a composition of flat morphisms is flat.

(3) Let $X \to Y$ be a morphism, $\mathcal{F}$ a quasi-coherent sheaf, $g\colon Z \to Y$ a morphism, and $p\colon X \times_Y Z \to X$ a projection. If $\mathcal{F}$ is flat over Y, then $p^*\mathcal{F}$ is flat over Z. In particular, a basechange of a flat morphism is flat.

(4) An open immersion is flat.

Proof. Follows easily from what we have already done. □

Theorem 1.6.3. *Let S, X, Y be locally noetherian schemes, and $f\colon X \to Y$ a morphism of schemes over S. Let $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Assume that all stalks of $\mathcal{F}$ are nonzero, $\mathcal{F}$ is flat over S, and for every $s \in S$ the pullback of $\mathcal{F}$ to X_s is flat over Y_s. Then $\mathcal{F}$ is flat over Y and Y is flat over S at all points $y \in f(X)$.*

Proof. Follows at once from theorem 1.5.4. □

Corollary 1.6.4. *Let S, X, Y be locally noetherian schemes. Let $f\colon X \to Y$ and $g\colon Y \to S$ be morphisms of schemes. If gf is flat and for every $s \in S$ the pullback $X_s \to Y_s$ of f is flat, then f is flat, and g is flat at all points $y \in f(X)$.*

2 Étale morphisms

2.1 The module of Kähler differentials

Definition 2.1.1. Let $A \to B$ be a morphism of rings, and M a B-module. An A-derivation $d\colon B \to M$ is an A-module morphism, which satsifies Leibnitz identity: $d(b_1b_2) = b_2d(b_1) + b_1d(b_2)$ for every $b_1, b_2 \in B$.

A sum of two derivations is again an A-derivation, as well as a scalar multiple of a derivation by an element of B. Hence, A-derivations $B \to M$ form a B-module, which is denoted $\mathrm{Der}_A(B, M)$. The association $M \mapsto \mathrm{Der}_A(B, M)$ is a covariant functor in an evident way.

Proposition 2.1.2. *Let $A \to B \to C$ be morphisms of rings. There is an induced exact sequence of functors*

$$0 \to \mathrm{Der}_B(C, -) \to \mathrm{Der}_A(C, -) \to \mathrm{Der}_A(B, {}^B-)$$

The first map takes a B-derivation and interprets it as an A-derivation. The second map precomposes a derivation with the morphism $B \to C$. The symbol ${}^B-$ denotes restriction of scalars from C to B.

Proof. The first map is obviously injective. If an A-derivation $d\colon C \to M$ vanishes when restricted to B, then it is a B-derivation, so the sequence is exact at $\mathrm{Der}_A(C, -)$. □

Proposition 2.1.3. *Let $A \to B$ be a morphism of rings, $I \subset B$ an ideal. There is an exact sequence of functors:*

$$0 \to \mathrm{Der}_A(B/I, -) \to \mathrm{Der}_A(B, {}^{B/I}-) \to \mathrm{Hom}_{B/I}(I/I^2, -)$$

The first map precomposes a derivation with $B \to B/I$. The second map restricts a derivation to I.

Proof. Let M be a B/I-module. Leibnitz identity and the fact that $IM = 0$ show that every A-derivation $d\colon B \to M$ vanishes on I^2, and determines a morphism $d\colon I/I^2 \to M$ of B/I-modules. On the other hand, if $d|_I = 0$, then clearly d comes from an A-derivation $B/I \to M$, hence the sequence is exact. □

Proposition 2.1.4. *Let $A \to B$ be a morphism of rings, $s \in B$ a unit, $b \in B$ an element, and $d\colon B \to M$ an A-derivation.*

$$d\Big(\frac{b}{s}\Big) = \frac{sdb - bds}{s^2}$$

Proof. From the formula $0 = d(1) = d(ss^{-1}) = sd(s^{-1}) + s^{-1}ds$ we conclude that $d\big(\frac{1}{s}\big) = -\frac{ds}{s^2}$, and then the claim follows by Leibnitz identity. □

Proposition 2.1.5. *Let A be a ring, $S \subset A$ a multiplicative system, and $A \to A_S$ a localization morphism. The functor $\mathrm{Der}_A(A_S, -)$ is zero.*

Proof. From the previous proposition it follows that every A-derivation of A_S is zero. □

Proposition 2.1.6. *Let $A \to B$ be a morphism of rings, and $S \subset B$ a multiplicative system. The morphism $\mathrm{Der}_A(B, {}^B-) \to \mathrm{Der}_A(B_S, -)$ induced by $B \to B_S$ is an isomoprhism.*

Proof. Let M be a B_S-module. We first show that the morphism in question is surjective. Let $d\colon B \to M$ be an A-derivation. It induces a derivation $D\colon B_S \to M$ by the rule

$$D\Big(\frac{b}{s}\Big) = \frac{sdb - bds}{s^2}.$$

Additivity and Leibnitz identity follow from trivial but lengthy calculations. Clearly, $D\big(\frac{b}{1}\big) = \frac{db}{1}$, so D is an A-derivation which restricts to d on B.

As for injectivity, consider the exact sequence of proposition 2.1.2 induced by $A \to B \to B_S$, and observe that $\mathrm{Der}_B(B_S, -) = 0$. □

Proposition 2.1.7. *Let A be a ring, let B, C be A-algebras. The morphism* $\operatorname{Der}_C(B \otimes_A C, -) \to \operatorname{Der}_A(B, {}^B-)$ *induced by ring morphism $B \to B \otimes_A C$ is an isomorphism.*

Proof. Let M be a module over $B \otimes_A C$. An element of $\operatorname{Der}_C(B \otimes_A C, M)$ is a bilinear map $d\colon B \times C \to M$ which satisfies the following identities for every $a \in A, b \in B, c \in C, b_i \in B$:

$$d(ab, c) = d(b, ac) = ad(b, c),$$
$$d(b, c) = (1 \otimes_A c)d(b, 1),$$
$$d(b_1 b_2, 1) = (b_1 \otimes_A 1)d(b_2, 1) + (b_2 \otimes_A 1)d(b_1, 1).$$

From this description it is clear that if d vanishes in $\operatorname{Der}_A(B, M)$, then $d = 0$. Given $D \in \operatorname{Der}_A(B, M)$ we define $d(b, c) = (1 \otimes_A c)D(b)$, which clearly satisfies the equation above, so the claim follows. □

Proposition 2.1.8. *Let $f\colon A \to B$ be a morphism of rings, $S \subset B$ a multiplicative system. The natural morphism* $\operatorname{Der}_{A_{f^{-1}S}}(B_S, -) \to \operatorname{Der}_A(B, -)$ *induced by ring morphisms $B \to B_S$ and $A \to A_{f^{-1}S}$ is an isomorphism.*

Proof. The morphism in question factors as

$$\operatorname{Der}_{A_{f^{-1}S}}(B_S, -) \to \operatorname{Der}_A(B_S, -) \to \operatorname{Der}_A(B, -).$$

Since $B_S \otimes_A A_{f^{-1}S} = B_S$, the first morphism is an isomorphism by proposition 2.1.7. The second morphism is an isomorphism by proposition 2.1.6. □

Theorem 2.1.9. *Let $A \to B$ be a morphism of rings. The functor* $\operatorname{Der}_A(B, -)$ *is representable.*

Proof. Such proofs are better done on one's own. □

Let $f\colon X \to Y$ be a morphism of schemes. One can extend the definition of Ω^1 to $X \to Y$ in two ways. First, since $\Omega^1_{B/A}$ commutes with restrictions to principal open subsets of $\operatorname{Spec} B$ and pullbacks to principal open subsets of $\operatorname{Spec} A$, one can pick a covering U_i of Y by open affines and coverings V_{ij} of $f^{-1}U_i$ by open affines, then glue various $\Omega^1_{V_{ij}/U_i}$, and show that this construction does not depend on the choice of covers. The other way is, given a morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces and a $\mathcal{O}_X$-module $\mathcal{F}$, define a $\mathcal{O}_X$-module of derivations $\operatorname{Der}_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$. One then shows that whenever X, Y are schemes and the morphism f is local, $\operatorname{Der}_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, -)$ is represented by a quasi-coherent $\mathcal{O}_X$-module, which agrees with Ω^1 when X and Y are affine. Either way, one obtains the following theorem:

Theorem 2.1.10. *To every morphism of schemes $f\colon X \to Y$ one can associate a quasi-coherent $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ which has following properties:*

- *If X, Y are affine, then $\Omega^1_{X/Y}$ coincides with the module of Kähler differentials associated to the ring morphism $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$.*
- *$\Omega^1_{X/Y}$ commutes with restrictions to opens $U \subset X$.*
- *Let $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ be morphism. The sheaf $\Omega^1_{X \times_S Y/Y}$ is isomorphic to $p^*\Omega^1_{X/S}$, where $p\colon X \times_S Y \to X$ is a projection.*
- *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms, then there is an exact sequence*
$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0.$$
- *If $X \xrightarrow{f} Y$ is a morphism and $Z \xrightarrow{g} X$ is a closed immersion with ideal sheaf $\mathcal{I}$, then there exists an exact sequence*
$$\mathcal{I}/\mathcal{I}^2 \to g^*\Omega^1_{X/Y} \to \Omega^1_{Z/Y} \to 0.$$
- *If $f\colon X \to Y$ is locally of finite type, then $\Omega^1_{X/Y}$ is locally of finite type (in particular, coherent if X is locally noetherian).*

2.2 Étale algebras over fields

Proposition 2.2.1. *Let $k \to K$ be a finite extension of fields. $\Omega^1_{K/k}$ vanishes if and only if $k \to K$ is separable.*

Proof. Assume that $k \to K$ is finite and separable. Let $x \in K$ be a primitive element, f its minimal polynomial. Let M be a K-module, and $d\colon K \to M$ a derivation.
$$0 = d(f(x)) = f'(x)dx.$$
Since K is separable, $f'(x) \neq 0$, so $dx = 0$ in M. Since K is generated over k by powers of x, we conclude that $d = 0$.

Assume that $k \to K$ is inseparable and primitive. Let $x \in K$ be a primitive element and f its minimal polynomial. Write $K = k[T]/(f)$. Recall that every derivation $d \in \mathrm{Der}_k(k[T], K)$ is determined by $d(T)$ and $d(T)$ can be arbitrary. Set $d(T) = x$. Then d vanishes when restricted to (f), since $d(gf) = g(x)f'(x)dx + f(x)dg = 0$ as $f(x) = 0$ and $f'(x) = 0$. Hence d comes from some derivation in $\mathrm{Der}_k(k[T]/(f), K)$ i.e. $\mathrm{Der}_k(K, K)$. As a consequence, $\mathrm{Der}_k(K, K) \neq 0$.

Assume that $k \to K$ is inseparable. There is a nontrivial proper subfield $E \subset K$ such that $E \to K$ is inseparable and primitive. Then $\Omega^1_{K/k}$ is nonzero, since its quotient $\Omega^1_{K/E}$ is nonzero. □

Proposition 2.2.2. *Let k be an algebraically closed field, A a k-algebra of finite type, and $\mathfrak{m} \in \operatorname{Specmax} A$. The homomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{A/k} \otimes_A k(\mathfrak{m})$ is an isomorphism.*

Proof. We need to prove that the natural restriction map

$$\operatorname{Der}_k(A, M) \to \operatorname{Hom}_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, M)$$

is an isomorphism for every $A/\mathfrak{m}$-module M.

By Hilbert's Nullstellensatz the composition $k \to A \to A/\mathfrak{m}$ is an isomorphism. In particular, $\operatorname{Der}_k(A/\mathfrak{m}, -) = 0$, so that the natural map in question is injective. Let $f\colon \mathfrak{m}/\mathfrak{m}^2 \to M$ be a morphism of $A/\mathfrak{m}$-modules. We define a map $d\colon A \to M$ by sending an element $a \in A$ to $f(a - a(\mathfrak{m}))$, where $a(\mathfrak{m})$ is the image of a modulo $\mathfrak{m}$ interpreted as an element of A. If $a_1, a_2 \in A$, then

$$a_1a_2 - a_1(\mathfrak{m})a_2(\mathfrak{m}) = \\ (a_1 - a_1(\mathfrak{m}))(a_2 - a_2(\mathfrak{m})) + a_2(\mathfrak{m})(a_1 - a_1(\mathfrak{m})) + a_1(\mathfrak{m})(a_2 - a_2(\mathfrak{m})).$$

Also, $a_i = a_i(\mathfrak{m})$ in $A/\mathfrak{m}$, so that $d(a_1a_2) = a_2d(a_1) + a_1d(a_2)$. Clearly, d vanishes on elements of k, so it is a derivation. □

Definition 2.2.3. Let k be a field. A k-algebra A is called étale if it is a finite cartesian product of finite separable extensions of k.

Theorem 2.2.4. *Let k be a field. A k-algebra of finite type A is étale if and only if $\Omega^1_{A/k} = 0$.*

Proof. Let A be a k-algebra of finite type such that $\Omega^1_{A/k} = 0$. Let us first assume that k is algebraically closed. By virtue of proposition 2.2.2 we then know that $\mathfrak{m}/\mathfrak{m}^2 = 0$ for every maximal ideal $\mathfrak{m}$ of A. Localizing at $\mathfrak{m}$ and applying Nakayama lemma we conclude that $A_\mathfrak{m}$ is a field, the kernel of the localization morphism $A \to A_\mathfrak{m}$ is $\mathfrak{m}$, and $A_\mathfrak{m} = A/\mathfrak{m}$. By Nullstellensatz, $A/\mathfrak{m} \cong k$.

Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime, and let $\mathfrak{m}$ be a maximal ideal containing it. Let $a \in \mathfrak{m}$. Since a vanishes in $A_\mathfrak{m}$, there exists $s \notin \mathfrak{m}$ such that $sa = 0$ in A. In particular, $sa \in \mathfrak{p}$, so $a \in \mathfrak{p}$. Hence each prime of A is maximal.

The algebra A is noetherian, so that the set of its minimal primes is finite. But all primes are maximal, so $\operatorname{Specmax} A$ is finite. Now, consider a morphism

$$A \to \prod_{\mathfrak{m} \in \operatorname{Specmax} A} A/\mathfrak{m} \tag{1}$$

By Chinese remainder theorem it is surjective. But $A/\mathfrak{m} = A_\mathfrak{m}$, so that the kernel of this morphism consists of elements which vanish in all localizations of A at maximal ideals, i.e. the kernel is zero. Hence, this morphism is an isomorphism. In particular, $\dim_k A$ is finite.

Now, let k be arbitrary, and $\overline{k}$ its algebraic closure. Let $A_{\overline{k}} = A \otimes_k \overline{k}$. Since $\dim_{\overline{k}} A_{\overline{k}}$ is finite, $\dim_k A$ is finite too. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime. The k-algebra $A/\mathfrak{p}$ is finite-dimensional and has no zero divisors, hence it is a field. So $\operatorname{Spec} A = \operatorname{Specmax} A$, and $\operatorname{Specmax} A$ is finite.

We consider a morphism as in (1). Its kernel is the nilradical of A. If $a \in A$ is nilpotent, then its image in $A_{\overline{k}}$ is nilpotent too, hence zero. But $A \to A_{\overline{k}}$ is injective, so that the kernel of (1) is zero. Now, proposition 2.2.1 finishes the proof. □

2.3 Unramified morphisms

Definition 2.3.1. Let $f\colon X \to Y$ be a morphism of schemes. We say that f is unramified if f is locally of finite type and $\Omega^1_{X/Y} = 0$.

Proposition 2.3.2. *Unramified morphisms have following properties:*

(1) If $f\colon X \to Y$ and $g\colon Y \to Z$ are unramified, then gf is unramified.

(2) If $f\colon X \to Y$ and $g\colon Y \to Z$ are such that gf is unramified, then f is unramified.

(3) If $f\colon X \to S$ is unramified, and $g\colon Y \to S$ is a morphism, then the pullback $X \times_S Y \to Y$ of f is unramified.

(4) Open immersions are unramified.

Proof. (1) The composition gf is locally of finite type. The exact sequence

$$g^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

implies that $\Omega^1_{X/Z} = 0$.

(2) The exact sequence above shows that $\Omega^1_{X/Y} = 0$. The fact that f is locally of finite type is left as an excercise (see [2] tag 01T8).

(3) Follows from proposition 2.1.7.

(4) Follows from proposition 2.1.5. □

Proposition 2.3.3. *Let $f\colon X \to Y$ be a morphism locally of finite type. It is unramified if and only if for each $y \in Y$ the fiber $X_y \to y$ is unramified.*

Proof. If $\Omega^1_{X/Y} = 0$, then clearly each fiber is unramified. Conversely, if $X_y \to y$ is unramified, then the fiber of $\Omega^1_{X/Y}$ at each point $x \in X$ is zero, as an inclusion of a point $x \in X$ factors through $X_{f(y)} \to X$. Since $\Omega^1_{X/Y}$ is locally of finite type, Nakayama lemma shows that $\Omega^1_{X/Y} = 0$. □

Proposition 2.3.4. *Let $X \to \operatorname{Spec} k$ be a scheme over a field. It is unramified if and only if X is discrete as a topological space, and for every $x \in X$ the field extension $k \to k(x)$ is finite separable.*

Proof. Assume that $X \to \operatorname{Spec} k$ is unramified. Let $x \in X$ and $U \subset X$ be an affine open neighbourhood of x which is of finite type over $\operatorname{Spec} k$. By theorem 2.2.4 we conclude that U is a spectrum of an étale algebra over k. In particular, U is discrete. Hence X is discrete.

Assuming the converse, take $x \in X$ and $U \subset X$ an affine open neighbourhood of x. Since X is discrete, U is discrete too, and as U is quasi-compact, we conclude that U is finite as a topological space. Hence U is a spectrum of an étale algebra over k, and so $\Omega^1_{X/k}|_U = 0$. As a consequence, $\Omega^1_{X/k} = 0$. Since U is a spectrum of an algebra of finite type over k, we conclude that $X \to \operatorname{Spec} k$ is locally of finite type. □

Proposition 2.3.5. *Let X, Y be schemes and $f \colon X \to Y$ a morphism locally of finite type. The fiber of $\Omega^1_{X/Y}$ at x is zero if and only if the residue field extension $k(f(x)) \to k(x)$ is finite separable, and $\mathfrak{m}_{Y,f(y)}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$.*

Proof. We immediately reduce to the case when $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine, and f is of finite type. Let $\mathfrak{q} \in \operatorname{Spec} B$ and $\mathfrak{p} = f(\mathfrak{q})$.

Assume that $\Omega^1_{B/A} \otimes_B k(\mathfrak{q}) = 0$. Since $\Omega^1_{B/A}$ is of finite type, Nakayama lemma implies that $(\Omega^1_{B/A})_{\mathfrak{q}} = 0$. Hence replacing B by its localization at some element not contained in $\mathfrak{q}$ we may assume that $\Omega^1_{B/A} = 0$. As a consequence, $\Omega^1_{B_{\mathfrak{p}}/A_{\mathfrak{p}}} = 0$.

Consider a ring $B \otimes_A k(\mathfrak{p})$. Since $\Omega^1_{B \otimes_A k(\mathfrak{p})/k(\mathfrak{p})} = 0$ and B is of finite type over A, theorem 2.2.4 shows that $B \otimes_A k(\mathfrak{p})$ is a finite étale algebra over $k(\mathfrak{p})$.

The morphism $A_{\mathfrak{p}} \to k(\mathfrak{p})$ is surjective, so $B_{\mathfrak{q}} \to B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$ is surjective. On the other hand

$$B \otimes_A k(\mathfrak{p}) = B \otimes_A (A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}),$$

so $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$ is a localization of a finite étale algebra over $k(\mathfrak{p})$, hence is itself such an algebra.

The morphism $\operatorname{Spec}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) \to \operatorname{Spec} B_{\mathfrak{q}}$ is a closed immersion. In particular, it is injective and sends closed points to closed points. As $B_{\mathfrak{q}}$ has only one maximal ideal, we conclude that $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$ also has unique maximal ideal, which forces it to be a finite separable field extension of $k(\mathfrak{p})$. On the other hand $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) = B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$, so that $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$.

Now, assume that $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$ is maximal, and that $k(\mathfrak{q})$ is a finite separable extension of $k(\mathfrak{p})$. Our assumptions imply that $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = B_\mathfrak{q} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) = k(\mathfrak{q})$. Hence $\Omega^1_{B/A} \otimes_B k(\mathfrak{q}) = \Omega^1_{B_\mathfrak{q}/A_\mathfrak{p}} \otimes_{B_\mathfrak{q}} k(\mathfrak{q}) = \Omega^1_{k(\mathfrak{q})/k(\mathfrak{p})} = 0$. □

2.4 Étale morphisms

Definition 2.4.1. Let $f\colon X \to Y$ be a morphism of schemes. We say that f is étale if it is unramified and flat.

Proposition 2.4.2. *Étale morphisms have following properties:*

(1) If $f\colon X \to Y$ and $g\colon Y \to Z$ are étale, then gf is étale.

(2) If $f\colon X \to S$ is étale, and $g\colon Y \to S$ is a morphism, then the pullback $X \times_S Y \to Y$ of f is étale.

(3) Open immersions are étale.

(4) If a morphism $f\colon X \to Y$ of schemes is locally of finite type, flat, and every fiber $X_y \to y$ is unramified, then f is étale.

Proof. Everything follows at once from corresponding properties of flat and unramified morphisms. □

Proposition 2.4.3. *Let $f\colon X \to Y$ and $g\colon Y \to S$ be morphisms of schemes. If gf is étale and g is unramified, then f is étale. If in addition f is surjective, then g is étale.*

Proof. Follows from corollary 1.6.4 because each fiber Y_s is a disjoint union of spectra of fields. □

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