Derived categories,
with a view toward coherent duality

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Contents

Throughout this text we will use notation, conventions and terminology of the Stacks project [2], with one exception. We allow arbitrary locally small categories besides the ones listed at [2] tag 0015.

As a general introduction to derived categories we recommend the books [1], and [4]. The notes of Alexander Kuznetsov [3] will be very useful for those who can read Russian.

1 Localization of categories

1.1 Localization

Definition 1.1.1. Let $\mathcal{C}$ be a category, and S a collection of morphisms. A localization of $\mathcal{C}$ at S is a pair $(\mathcal{C}_S, Q)$, where $\mathcal{C}_S$ is a category, and $Q\colon \mathcal{C} \to \mathcal{C}_S$ a functor, with the following properties.

(1) Q sends all morphisms from S to isomorphisms.

(2) If $\mathcal{D}$ is a category, and $F\colon \mathcal{C} \to \mathcal{D}$ a functor such that $F(s)$ is an isomorphism for every s from S, then there exists a pair (F_S, η), where $F_S\colon \mathcal{C}_S \to \mathcal{D}$ is a functor, and $\eta\colon F \to F_S Q$ is an isomorphism.

(3) Given two pairs (F_S, η), (F'_S, η') as above there exists a unique natural transformation $\xi\colon F_S \to F'_S$ such that $\xi_Q \eta = \eta'$.

Unicity in (3) forces ξ to be an isomorphism.

Proposition 1.1.2. *Let $\mathcal{C}$ be a category, S a collection of morphisms. If $(\mathcal{C}_S, Q)$ and $(\mathcal{C}'_S, Q')$ are localizations of $\mathcal{C}$ at S, then $\mathcal{C}_S$ and $\mathcal{C}'_S$ are equivalent.*

Proof. Since Q' transforms morphisms from S to isomorphisms, there exists a functor $F\colon \mathcal{C}_S \to \mathcal{C}'_S$ and a natural isomorphism $\eta'\colon Q' \to FQ$. Similarly, there exists a functor $G\colon \mathcal{C}'_S \to \mathcal{C}_S$ and a natural isomorphism $\eta\colon Q \to GQ'$. Thus we have an isomorphism $G(\eta')\eta\colon Q \to GFQ$, and an an isomorphism $1_Q\colon Q \to 1_{\mathcal{C}_S} Q$. By property (3) of localizations there exists a unique natural isomorphism $\mu\colon 1_{\mathcal{C}_S} \to GF$ such that $\mu_Q = G(\eta')\eta$. Similarly, there exists a unique natural isomorphism $\nu\colon 1_{\mathcal{C}'_S} \to FG$ such that $\nu_{Q'} = F(\eta)\eta'$. □

Proposition 1.1.3. *Let $\mathcal{C}$ be a category, S a collection of morphisms, $(\mathcal{C}_S, Q)$ a localization of $\mathcal{C}$ at S. Let $\mathcal{C}'_S$ be another category. If a functor $F\colon \mathcal{C}_S \to \mathcal{C}'_S$ defines an equivalence of $\mathcal{C}_S$ and $\mathcal{C}'_S$, then $(\mathcal{C}'_S, FQ)$ is a localization of $\mathcal{C}$ at S.*

Proof. Property (1) of localization is clear.

Let $G\colon \mathcal{C}'_S \to \mathcal{C}_S$ be a functor, and $\mu\colon 1_{\mathcal{C}_S} \to GF$, $\nu\colon FG \to 1_{\mathcal{C}'_S}$ natural isomorphisms. We can assume that μ and ν make F left adjoint to G. Indeed, we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}'_S}(F-,-) \xleftarrow{(\nu)_*} \mathrm{Hom}_{\mathcal{C}'_S}(F-,FG-) \xleftarrow{F} \mathrm{Hom}_{\mathcal{C}_S}(-,G-).$$

Let $\mathcal{D}$ be a category, and $H\colon \mathcal{C} \to \mathcal{D}$ a functor such that $H(s)$ is an isomorphism for every s from S. There exists a functor $H_S\colon \mathcal{C}_S \to \mathcal{D}$ and a natural isomorphism $\eta\colon H \to H_S Q$. Consider a functor $H_S G\colon \mathcal{C}'_S \to \mathcal{D}$. Composing the natural isomorphism η with the isomorphism $H_S(\mu_Q)$ we obtain an isomorphism of H and $(H_S G)(FQ)$. Hence the property (2) also holds.

Let $H_i\colon \mathcal{C}'_S \to \mathcal{D}$, $i = 1,2$ be functors, and $\eta_i\colon H \to H_i FQ$ natural isomorphisms. There exists a unique morphism $\xi\colon H_1F \to H_2F$ such that $\xi_Q\eta_1 = \eta_2$. First, let us show that if there is a natural transformation $\theta\colon H_1 \to H_2$, such that $\theta_{FQ}\eta_1 = \eta_2$, then it is unique. Indeed, $\theta_F = \xi$ by unicity of ξ. The square

$$\begin{array}{ccc} H_1 & \xrightarrow{\theta} & H_2 \\ \uparrow{\scriptstyle H_1(\nu)} & & \uparrow{\scriptstyle H_2(\nu)} \\ H_1FG & \xrightarrow{\theta_{FG}} & H_2FG \end{array}$$

commutes by naturality of θ. Hence θ is determined by ξ_G, and so is unique.

Next, we define θ by a diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{\theta} & H_2 \\ \uparrow{\scriptstyle H_1(\nu)} & & \uparrow{\scriptstyle H_2(\nu)} \\ H_1FG & \xrightarrow{\xi_G} & H_2FG. \end{array}$$

Consider a diagram

$$\begin{array}{ccc} H_1FQ & \xrightarrow{\theta_{FQ}} & H_2FQ \\ \uparrow{\scriptstyle H_1(\nu)_{FQ}} & & \uparrow{\scriptstyle H_2(\nu)_{FQ}} \\ H_1FGFQ & \xrightarrow{\xi_{GFQ}} & H_2FGFQ \\ \uparrow{\scriptstyle H_1F(\mu_Q)} & & \uparrow{\scriptstyle H_2F(\mu_Q)} \\ H_1FQ & \xrightarrow{\xi_Q} & H_2FQ. \end{array}$$

Since F is left adjoint to G, the vertical composite arrows in this diagram are identities. $\square$

Proposition 1.1.4. *Let $\mathcal{C}$ be a category, S a collection of morphisms. If S consists of isomorphisms then $(\mathcal{C}, \mathrm{Id}_{\mathcal{C}})$ is a localization of $\mathcal{C}$ at S.*

Proof. Omitted. $\square$

1.2 Derived category as a localization

Let $\mathcal{A}$ be an abelian category. Consider its homotopy category $K(\mathcal{A})$. Recall that a morphism of complexes $s\colon A_1 \to A_2$ is called a quasi-isomorphism if the induced maps on cohomology $H^n(s)$ are isomorphisms.

Definition 1.2.1. The derived category $D(\mathcal{A})$ is the localization of $K(\mathcal{A})$ at the collection of all quasi-isomorphisms.

Since the cohomology functors $H^n\colon K(\mathcal{A}) \to \mathcal{A}$ send quasi-isomorphisms to isomorphisms by construction, we obtain induced functors $H^n\colon D(\mathcal{A}) \to \mathcal{A}$. Unlike cohomology, the terms of a complex $A \in \mathrm{Comp}(\mathcal{A})$ can not be extracted from it in $D(\mathcal{A})$ or even $K(\mathcal{A})$.

The category $K(\mathcal{A})$ has as full subcategories the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$ of complexes bounded below, above, or on both sides respectively. We define $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ as localizations of respective categories at quasi-isomorphisms. One can check that $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ are full subcategories of $D(\mathcal{A})$, and that $D^b(\mathcal{A})$ is a full subcategory of $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ provided all relevant categories exist (see [2] tag 05RV). The categories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ are easier to handle than the unbounded category $D(\mathcal{A})$.

It is not clear a priori that $D(\mathcal{A})$ exists. If $\mathcal{A}$ is (equivalent to) a small category then we will see below that $D(\mathcal{A})$ exists. Furthermore, $D(\mathcal{A})$ exists if $\mathcal{A}$ is a Grothendieck abelian category, for example the category of modules over a ring, (pre)sheaves on a site, $\mathcal{O}_X$-modules on a ringed site, or quasi-coherent sheaves on a scheme[1] (see [2] tag 03EW). In the following we will show that $D(\mathcal{A})$ exists when $\mathcal{A}$ is (equivalent to) a small category, $D^+(\mathcal{A})$ exists if $\mathcal{A}$ has enough injectives, and $D^-(\mathcal{A})$ exists if $\mathcal{A}$ has enough projectives.

If the category $\mathcal{A}$ is semisimple then one can construct $D(\mathcal{A})$ explicitly. Recall that an abelian category $\mathcal{A}$ is semisimple if every monomorphism (equivalently, every epimorphism) in it splits. Examples of such categories are vector spaces over a field, or representations of a finite group over a field of characteristic 0.

Let A be a complex in $\mathcal{A}$. Since $\mathcal{A}$ is semisimple, each object A^n can be noncanonically decomposed as $A^n = B^n \oplus H^n \oplus E^n$, where B^n is the image of A^{n-1} under d^{n-1}, $B^n \oplus H^n$ is the kernel of d^n, and H^n is mapped isomorphically to $H^n(A)$. The differential $d^n\colon A^n \to A^{n+1}$ sends an element $(b, h, e) \in B^n \oplus H^n \oplus E^n$ to $(d^n e, 0, 0)$. It follows that E^n is mapped by d^n isomorphically to B^{n+1}. Moreover, A as a complex is isomorphic to $H \oplus G$ where H has zero differential, $G^n = B^n \oplus E^n$, and d^n_G sends (b, e) to $(d^n e, 0)$.

We next claim that G is zero in $K(\mathcal{A})$. Consider a homotopy $u^n\colon G^n \to G^{n-1}$ which sends $(b, e) \in B^n \oplus E^n$ to $(0, e')$ where e' is the preimage of b in E^{n-1}

[1]The fact that the category of quasi-coherent sheaves on an arbitrary scheme is Grothendieck is due to Gabber. See [2] tag 077K for an exposition.

under d^n. Clearly $u^{n+1}d^n_G + d^{n-1}_G u^n$ is the identity map, so id_G is homotopy equivalent to zero in $K(\mathcal{A})$, which means precisely that $G = 0$. Therefore A is isomorphic to H in $K(\mathcal{A})$.

Let $f\colon A_1 \to A_2$ be a morphism of complexes. Consider splittings $A_i = B^n_i \oplus H^n_i \oplus E^n_i$ as above. Since B^n_1 is the image of A^{n-1}_1 under d^{n-1}, and f commutes with d^n, it maps B^n_1 to B^n_2. Similarly, since $B^n_1 \oplus H^n_1$ is the kernel of d^n, f maps it to $B^n_2 \oplus H^n_2$. Therefore f maps H^n_1 to H^n_2. We conclude that f splits into a direct sum of morphisms $f_H\colon H_1 \to H_2$ and $f_G\colon G_1 \to G_2$.

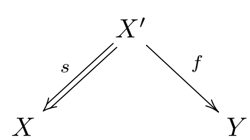
If $s\colon A_1 \to A_2$ is a quasi-isomorphism, then the induced map $H_1 \to H_2$ is an isomorphism, because H^n_i is functorially isomorphic to $H^n(A_i)$. Since $A_i \cong H_i$ in $K(\mathcal{A})$ we conclude that every quasi-isomorphism in $K(\mathcal{A})$ is an isomorphism. Thus $D(\mathcal{A}) \cong K(\mathcal{A})$ with the quotient functor being identity.

More precisely, one can consider the full subcategory $K(\mathcal{A})_0 \subset K(\mathcal{A})$ consisting of complexes with differential zero. Notice that $K(\mathcal{A})_0$ is a countable product of copies of $\mathcal{A}$ indexed by $n \in \mathbf{Z}$. Sending a complex A to its cohomology viewed as a complex with differential zero defines a functor $Q\colon K(\mathcal{A}) \to K(\mathcal{A})_0$. By the discussion above Q is an equivalence of categories. As a consequence $D(\mathcal{A}) \cong \prod_{n\in\mathbf{Z}} \mathcal{A}$ with the quotient functor being Q.

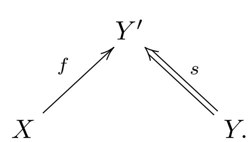
2 Calculus of fractions

Let $\mathcal{C}$ be a category, S a collection of morphisms in $\mathcal{C}$. In diagrams below we will use double arrows $\Rightarrow$ to symbolise the fact that respective morphisms belong to S.

Definition 2.0.2. Let X, Y objects of $\mathcal{C}$. A right fraction from X to Y is a diagram



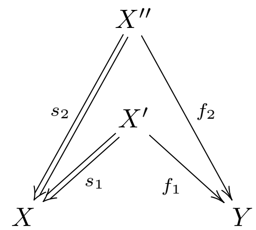
Dually, a left fraction from X to Y is a diagram



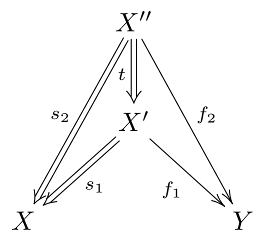
Unfortunately in the literature there is no agreement on which fractions are left, and which are right. We follow conventions of Stacks Project.

In general the collection of fractions from X to Y is not a set but only a proper class.

Definition 2.0.3. Consider a pair of right fractions:

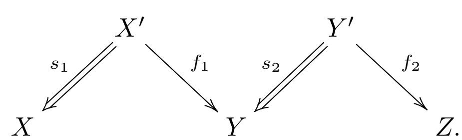


We say that the outer fraction dominates the inner one if there is an arrow $t\colon X'' \Rightarrow X'$ from S such that the diagram



is commutative.

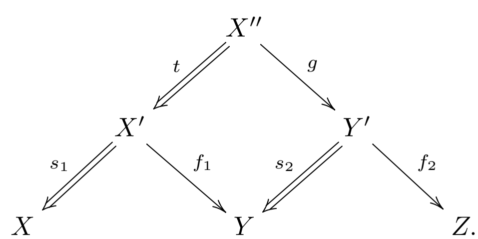
Definition 2.0.4. Consider a pair of right fractions:



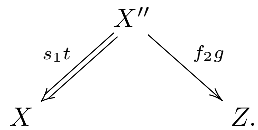
(1)

The fractions in such a diagram are called composable.

We say that composition of fractions in (1) is defined if the diagram (1) can be completed to a commutative diagram

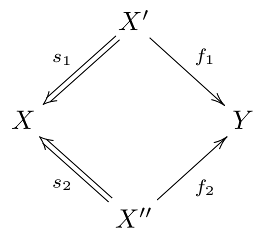


The outer fraction

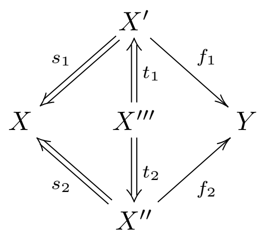


is called the composition of fractions in (1).

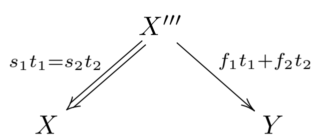
Definition 2.0.5. Suppose that the category $\mathcal{C}$ is preadditive. Consider a pair of right fractions



If there exists an object X''' and morphisms $t_1 \colon X''' \Rightarrow X'$, $t_2 \colon X''' \Rightarrow X''$ such that the diagram



commutes, then we say that the right fraction



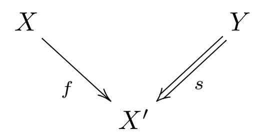
is the sum of fractions in (2.0.5).

Relations and operations on left fractions are defined in the same way.

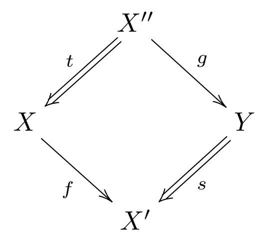
Definition 2.0.6. Let $\mathcal{C}$ be a category, and S a collection of morphisms. We say that S is satisfies right Ore conditions if the following properties hold.

(2-out-of-3) Every isomorphism is in S, and if in an expression $fg = h$ two morphisms belong to S then so is the third.

(RMS2) Every *left* fraction



can be completed to a commutative square

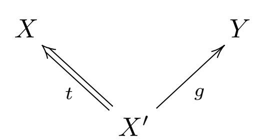


in which t belongs to S.

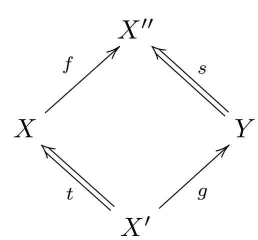
(RMS3) If for a pair of morphisms $f, g\colon X \to Y$ there exists a morphism $s\colon Y \Rightarrow Y'$ from S such that $sf = sg$, then there exists a morphism $t\colon X' \Rightarrow X$ from S such that $ft = gt$.

Definition 2.0.7. Let $\mathcal{C}$ be a category, and S a collection of morphisms. We say that S is satisfies left Ore conditions if the 2-out-of-3 property holds, as well as the following properties.

(LMS2) Every *right* fraction



can be completed to a commutative square

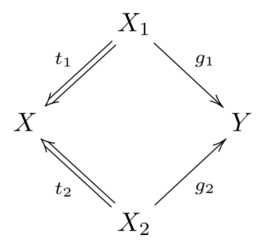


in which s belongs to S.

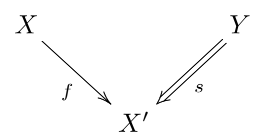
(LMS3) If for a pair of morphisms $f, g\colon X \to Y$ there exists a morphism $t\colon X' \Rightarrow X$ from S such that $ft = gt$, then there exists a morphism $s\colon Y \Rightarrow Y'$ from S such that $sf = sg$.

Proposition 2.0.8. *Let $\mathcal{C}$ be a category, and S a collection of morphisms satisfying right Ore conditions. Consider a pair of right fractions*

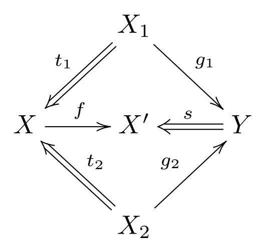
(2)



If there exists a left fraction



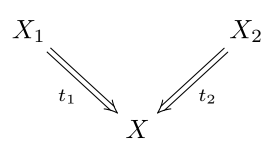
such that the diagram



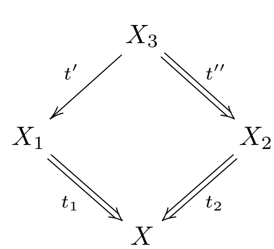
is commutative, then there exists a right fraction dominating both fractions in (2)*.*

If S satisfies left Ore conditions then the same statement holds with the roles of left and right fractions interchanged.

Proof. First complete the left fraction

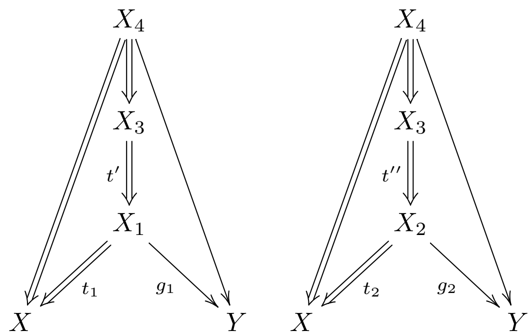


to a diagram



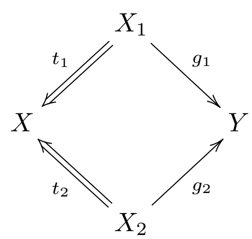
The morphism t' belongs to S by 2-out-of-3 property.

Notice that $sg_1t' = ft_1t' = ft_2t'' = sg_2t''$. RMS3 implies that there exists a morphism $t''' \colon X_4 \Rightarrow X_3$ from S such that $g_1t't''' = g_2t''t'''$. Thus the outer fractions in the diagrams



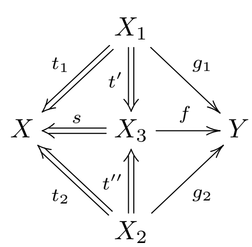
coincide. By construction this outer fraction dominates both fractions in (2). □

Proposition 2.0.9. *Let $\mathcal{C}$ be a category, and S a collection of morphisms satisfying right Ore conditions. Consider a pair of right fractions*



(3)

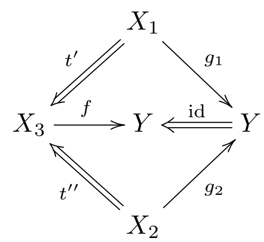
Suppose that they dominate another right fraction, i.e. that there is a commutative diagram



Then there exists a right fraction dominating both fractions in (3).

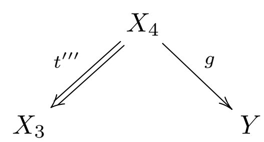
If S satisfies left Ore conditions then left fractions enjoy the same property.

Proof. Consider a diagram

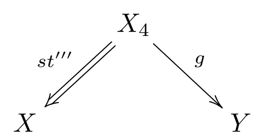


(4)

Proposition 2.0.8 shows that there exists a right fraction



dominating both right fractions in (4). Then the right fraction



dominates both right fractions in (3). ∎

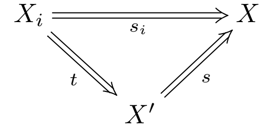
Definition 2.0.10. We say that two right fractions from X to Y are equivalent if there exists a third one dominating both of them. Similarly for left fractions.

The relation on fractions defined above is obviously reflexive and symmetric. Proposition 2.0.9 shows that it is also transitive when S satisfies right (respectively, left) Ore conditions.

As it was mentioned earlier, the collection of all right (left) fractions from X to Y does not form a set in general. Let us introduce definitions to deal with this problem.

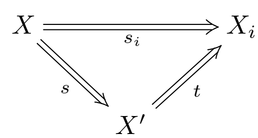
Definition 2.0.11. Let $\mathcal{C}$ be a category, and S a collection of morphisms. Let X be an object of $\mathcal{C}$.

(1) We say that S is locally small on the right at X if there exists a *set* of morphisms $S_X = \{s_i\colon X_i \Rightarrow X\}$ from S such that whenever $s\colon X' \Rightarrow X$ is a morphism in S then there exists a morphism $s_i\colon X_i \Rightarrow X$ from S_X, and a morphism $t\colon X_i \Rightarrow X'$ from S such that the diagram



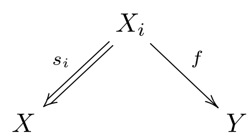
is commutative. We say that S is locally small on the right if it is locally small on the right at every $X \in \mathcal{C}$.

(2) We say that S is locally small on the left at X if there exists a *set* of morphisms $S_X = \{s_i : X \Rightarrow X_i\}$ from S such that whenever $s : X \Rightarrow X'$ is a morphism in S then there exists a morphism $s_i : X \Rightarrow X_i$ from S_X, and a morphism $t : X' \Rightarrow X_i$ from S such that the diagram



is commutative. We say that S is locally small on the left if it is locally small on the left at every $X \in \mathcal{C}$.

Proposition 2.0.12. *Let $\mathcal{C}$ be a category, S a collection of morphisms, and X, Y objects of $\mathcal{C}$. Suppose that S is locally small on the right at X. Let $S_X = \{s_i : X_i \Rightarrow X\}$ be a set of morphisms as in definition 2.0.11. Let $F_{X,Y}$ be the set of all right fractions of the form*



where s_i runs over S_X, and f runs over $\mathrm{Hom}_{\mathcal{C}}(X_i, Y)$.

(1) *Every right fraction from X to Y is dominated by one from $F_{X,Y}$.*

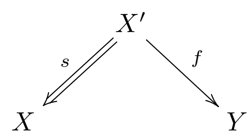
(2) *If S satisfies right Ore conditions then the relation on $F_{X,Y}$ induced from the relation on the collection of all fractions from X to Y is reflexive, symmetric, and transitive. Hence we obtain a set of equivalence classes $F_{X,Y}/\sim$.*

(3) *$F_{X,Y}/\sim$ is independent of the choice of S_X in the following sense.*

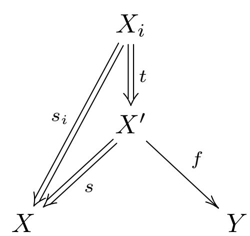
If $F'_{X,Y}$ is any set of right fractions from X to Y then the equivalence relation on all fractions from X to Y induces an equivalence relation $\sim$ on $F'_{X,Y}$. There is a unique map $\varphi : F'_{X,Y}/\sim \to F_{X,Y}/\sim$ characterized by the property that it sends an equivalence class represented by a fraction $f \in F'_{X,Y}$ to an equivalence class containing a fraction that dominates f.

Now, if we start from a set of morphisms S'_X satisfying conditions of definition 2.0.11 and construct a set of fractions $F'_{X,Y}$ as in (1) then the map $\varphi : F'_{X,Y}/\sim \to F_{X,Y}/\sim$ described above is a bijection.

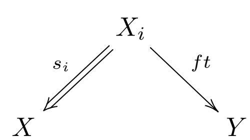
Proof. (1) Indeed, every right fraction



fits into a commutative diagram



for some $s_i\colon X_i \Rightarrow X$ from S_X, and some $t\colon X_i \Rightarrow X'$. The fraction



belongs to $F_{X,Y}$ and dominates the initial fraction by construction.

(2) Transitivity of the induced relation follows at once from proposition (2.0.9).

(3) Omitted. $\square$

An analogous proposition holds for left fractions, and collections S which are small on the left.

Definition 2.0.13. In the following we will say that a class $C \in F_{X,Y}/\sim$ dominates a fraction if C contains a fraction dominating the given one.

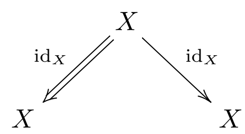
Theorem 2.0.14. *Let $\mathcal{C}$ be a category, and S a class of morphisms.*

(1) *Suppose that S satisfies right Ore conditions, and is locally small on the right. Consider a category $\mathcal{C}_S$ defined as follows.*

(1a) *Objects of $\mathcal{C}_S$ are objects of $\mathcal{C}$.*

(1b) *For two objects $X, Y \in \mathcal{C}_S$ the set of morphisms $\mathrm{Hom}_{\mathcal{C}_S}(X,Y)$ is defined to be equal to the set $F_{X,Y}/\sim$ constructed in proposition 2.0.12 (2).*

(1c) *For an object $X \in \mathcal{C}_S$ we define the identity morphism in* $\mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ *to be the class dominating the fraction*



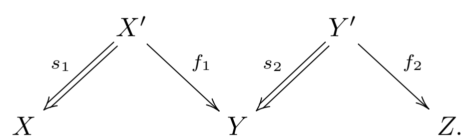
(5)

(1d) *Given three objects $X, Y, Z \in \mathcal{C}_S$ and classes of fractions $f \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}_S}(Y, Z)$ we define the composition gf as follows. We pick arbitrary representatives from f and g and set gf to be the class of fractions from $\mathrm{Hom}_{\mathcal{C}_S}(X, Z)$ dominating the composition of representatives as described in definition 2.0.4.*

The category $\mathcal{C}_S$ is well-defined (i.e. satisfies category axioms), and locally small.

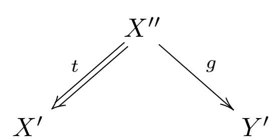
(2) *Suppose that S satisfies left Ore conditions, and is locally small on the left. Then analogously to (1) we define a category $\mathcal{C}_S$ using left fractions. This category is again well-defined an locally small.*

Proof. (1) Consider a pair of composable fractions



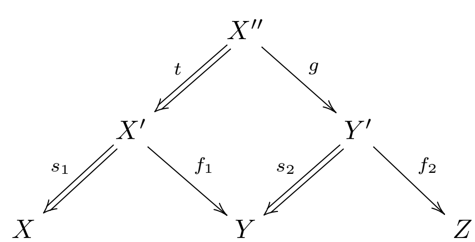
(6)

Since S satisfies right Ore conditions, there exists a right fraction



(7)

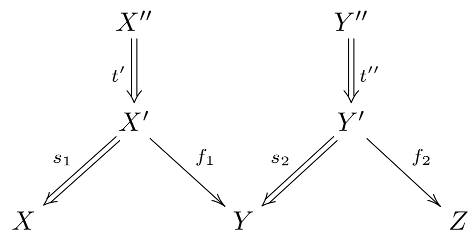
such that the diagram



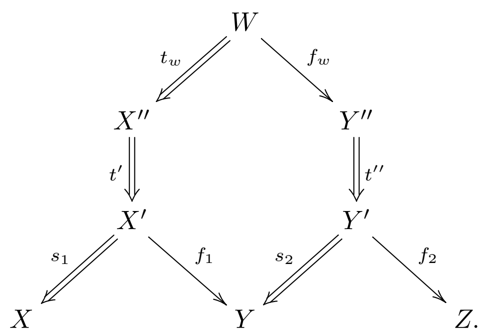
(8)

is commutative. Proposition 2.0.8 shows that equivalence class of the outer fraction in this diagram is independent of the choice of completing fraction (7).

To show that equivalence class of the composition is idepended of the choice of representatives it is enough to check that this class does not change when we replace the fractions in (6) by dominating ones. So, suppose that given a diagram

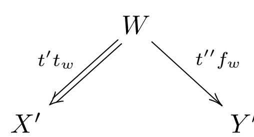


we completed it to the diagram



(9)

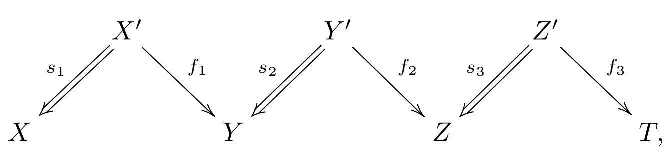
The fraction



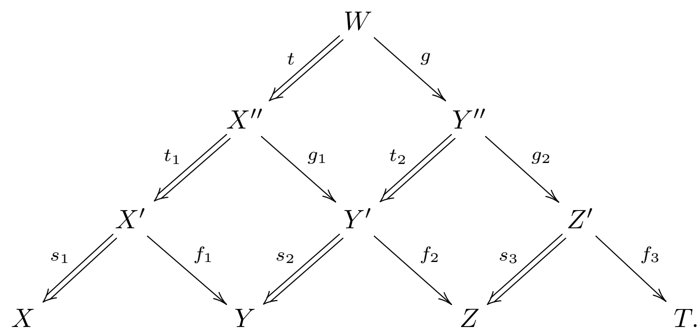
(10)

also completes the diagram (6). Proposition 2.0.8 shows that the fraction (10) is equivalent to (7), so the outer fraction in (9) is equivalent to that in (8). Thus the composition operation is well-defined.

Demonstrating that it is associative is not hard at all. Consider three composable fractions

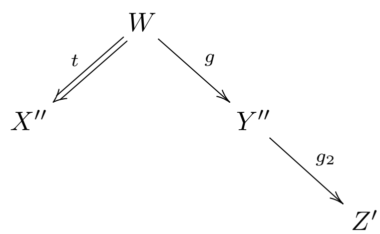


which we will denote f, g and h (from left to right). We complete them to a diagram

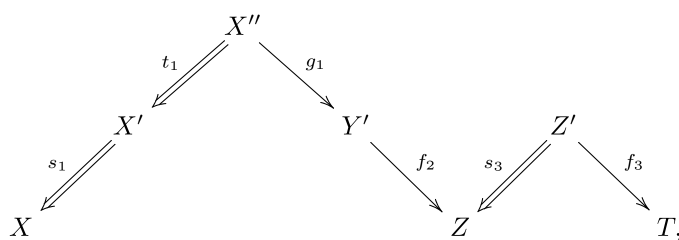


(11)

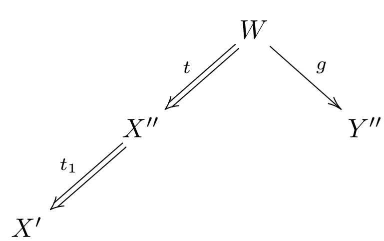
Now, the fraction



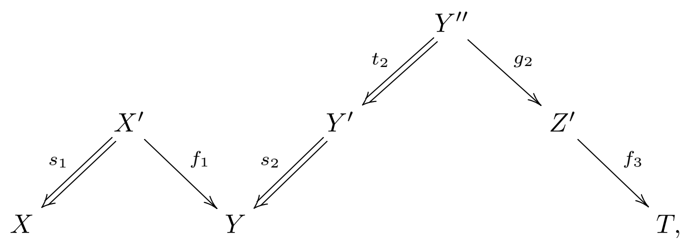
completes the diagram



so that the outer fraction in (11) represents the composition $h \circ (g \circ f)$. On the other hand, the fraction



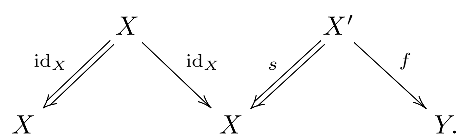
completes the diagram



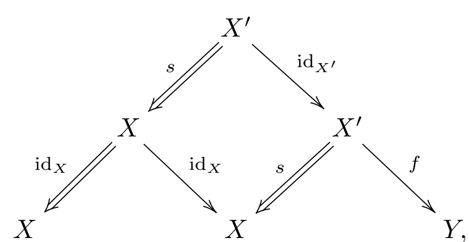
so that the outer fraction in (11) represents the composition $(h \circ g) \circ f$.

It remains to check that our composition operation respects identity morphisms. To do it it is enough to calculate the composition of fraction (5) itself with an arbitrary fraction, since our operation respects equivalence of fractions.

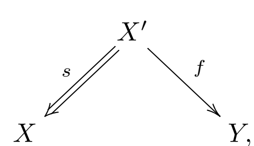
Consider a diagram



It can be completed to a diagram



so that the composition is represented by a fraction



which is precisely what we need. Composition in the other order is done similarly.

(2) The proof for left fractions is analogous. □

Proposition 2.0.15. *Let $\mathcal{C}$ be a preadditive category, and S a class of morphisms.*

(1) *Suppose that S satisfies right Ore conditions, and is locally small on the right. Consider the category $\mathcal{C}_S$ constructed in theorem 2.0.14 (1). Define addition of morphisms in it as follows.*

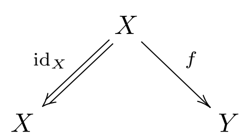
Let $X, Y \in \mathcal{C}_S$, and let $f, g \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$. We pick representatives from f and g, and construct their sum as described in the definition 2.0.5. We then let $f + g$ to be the class from $\mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ dominating the constructed sum.

The category $\mathcal{C}_S$ with this addition structure is preadditive.

(2) *If S satisfies left Ore conditions, and is locally small on the left then the category $\mathcal{C}_S$ constructed in theorem 2.0.14 (2) has a preadditive structure defined in a similar way.*

Proof. Left as an excercise. □

Theorem 2.0.16. *Let $\mathcal{C}$ be a category, and S a class of morphisms. Suppose that S satisfies right Ore conditions, and is locally small on the right. Consider a functor $Q\colon \mathcal{C} \to \mathcal{C}_S$ which sends an object $X \in \mathcal{C}$ to itself, and a morphism $f\colon X \to Y$ to the class of fractions from X to Y dominating the fraction*



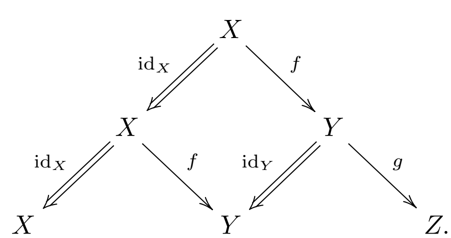
(1) *The functor Q is well-defined.*

(2) *The pair $(\mathcal{C}_S, Q)$ is a localization of $\mathcal{C}$ at S.*

(3) *If $\mathcal{C}$ is preadditive then Q is additive. If $\mathcal{C}$ is additive, then so is $\mathcal{C}_S$.*

(4) *An analogous statement is true for left fractions.*

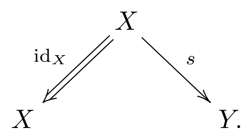
Proof. (1) $Q(\mathrm{id}) = \mathrm{id}$ by construction. Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms. Consider a diagram



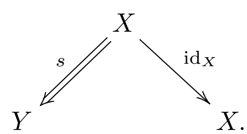
This diagram commutes, so $Q(g)Q(f)$ is equal to the class dominating the outer fraction in this diagram. But this fraction is precisely $Q(gf)$. So, Q commutes with composition of morphisms.

(2) We split the proof in three parts verifying conditions (1), (2), and (3) of definition 1.1.1.

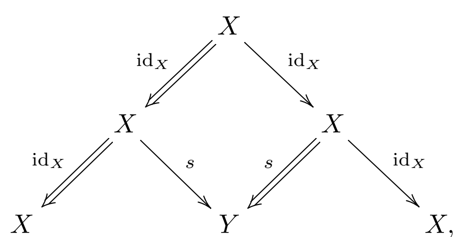
(2.1) We first check that Q sends S to isomorphisms. Consider a fraction



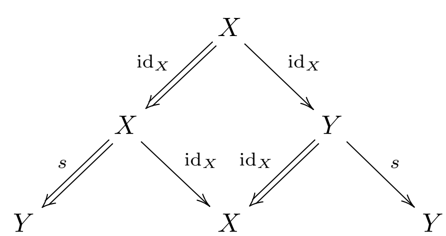
with s from S. We claim that the fraction



is its inverse. Indeed both diagrams

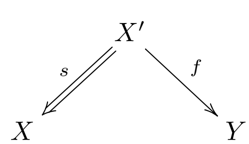


and

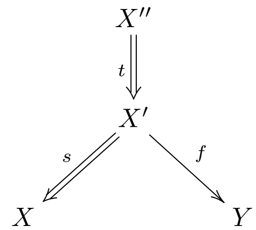


are commutative. The outer fraction in the first diagram is just the identity fraction, and the outer fraction in the second diagram dominates the identity fraction $Y \to Y$ via $s\colon X \to Y$.

(2.2) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which sends all morphisms from S to isomorphisms. Define $F_S\colon \mathcal{C}_S \to \mathcal{D}$ by sending an object $X \in \mathcal{C}_S$ to $F(X)$, and a fraction

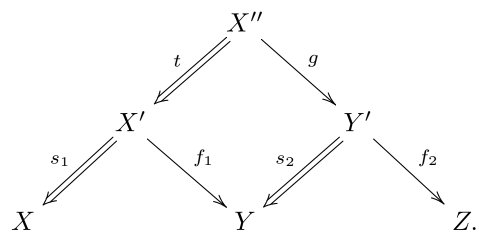


to $F(f)F(s)^{-1}$. First of all, F_S is well-defined on morphisms, since if we replace a fraction above by a dominating fraction



then $F(f)F(s)^{-1}$ is replaced by $F(ft)F(st)^{-1} = F(f)F(t)F(t)^{-1}F(s)^{-1} = F(f)F(s)^{-1}$.

Consider a composition diagram

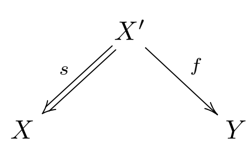


Since the square in the middle commutes, we obtain an equality $F(s_2)^{-1}F(f_1) = F(g)F(t)^{-1}$, so that $F(f_2g)F(s_1t)^{-1} = F(f_2)F(s_2)^{-1}F(f_1)F(s_1)^{-1}$. Thus F_S is a functor. The composition F_SQ sends an object $X \in \mathcal{C}_S$ to $F(X)$, and a morphism $f\colon X \to Y$ to $F(f)$, which means that we have a natural isomorphism $F \to F_SQ$.

(2.3) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which sends all morphisms from S to isomorphisms, $F_1, F_2\colon \mathcal{C}_S \to \mathcal{D}$ functors, and $\eta_i\colon F \to F_iQ$ natural isomorphisms.

To define a natural transformation $\xi\colon F_1 \to F_2$ notice that the equation $\xi_Q \circ \eta_1 = \eta_2$ and the fact that Q is surjective on objects forces us to set $\xi_X = \eta_{2,X} \circ \eta_{1,X}^{-1}$. So the natural transformation $\xi\colon F_1 \to F_2$ is unique, and we only need to check that the maps $\xi_X = \eta_{2,X} \circ \eta_{1,X}^{-1}$ indeed give a natural transformation.

Since η_i is a natural isomorphism $F_iQ(s)$ is an isomorphism for every s from S. Next, notice that a fraction



(12)

is tautologically a composition of fractions

$$X \xLeftarrow{\;s\;} X' \xrightarrow{\mathrm{id}_{X'}} X' \xLeftarrow{\mathrm{id}_{X'}} X' \xrightarrow{\;f\;} Y.$$

The fraction on the left is $Q(s)^{-1}$ and the fraction on the right is $Q(f)$. Thus F_i of the fraction (12) is equal to $F_iQ(f) \circ F_iQ(s)^{-1}$.

Now the commutative diagram

$$\begin{array}{ccccc}
F_1(X) & \xleftarrow{F_1Q(s)} & F_1(X') & \xrightarrow{F_1Q(f)} & F_1(Y) \\
\uparrow{\scriptstyle \eta_{1,X}} & & \uparrow{\scriptstyle \eta_{1,X'}} & & \uparrow{\scriptstyle \eta_{1,Y}} \\
F(X) & \xleftarrow{F(s)} & F(X') & \xrightarrow{F(f)} & F(Y) \\
\downarrow{\scriptstyle \eta_{2,X}} & & \downarrow{\scriptstyle \eta_{2,X'}} & & \downarrow{\scriptstyle \eta_{2,Y}} \\
F_2(X) & \xleftarrow{F_2Q(s)} & F_2(X') & \xrightarrow{F_2Q(f)} & F_2(Y)
\end{array}$$

shows that ξ is indeed a natural transformation. Therefore $(\mathcal{C}_S, Q)$ is a localization of $\mathcal{C}$ at S.

The rest of the proof is left as an excercise. □

3 The homotopy category

3.1 Cone

For each $q \in \mathbf{Z}$ we define a translation functor $[q]\colon \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A})$ as follows. If A is an object of $\mathrm{Comp}(\mathcal{A})$, then

$$A[q]^n = A^{n+q}, \quad d^n_{A[q]} = (-1)^q d^{n+1}_A.$$

If $f\colon A \to B$ is a morphism, then

$$(f[q])^n = f^{n+q}.$$

By construction, $[0]$ is the identity functor, $[q''] \circ [q'] = [q'' + q']$, $H^n(A[q]) = H^{n+q}(A)$, and $H^n(f[q]) = H^{n+q}(f)$.

Note that $[q]$ alters the sign of the differential. Beware that in some texts, notably in [4], the complex is shifted in opposite direction.

Let $A \xrightarrow{f} B$ be a morphism of complexes. The cone of f is the complex $A[1] \oplus B$ with the differential

$$\begin{pmatrix} d_{A[1]} & 0 \\ f & d_B \end{pmatrix}.$$

Since [1] alters the sign of d_A, the square of this matrix is zero, and we indeed get a complex.

The cone of f fits into a short exact sequence of complexes:

$$0 \longrightarrow B \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{p} A[1] \longrightarrow 0, \tag{13}$$

where p is the inclusion of B as a direct summand to $A[1] \oplus B$, and p is the projection. This short exact sequence gives rise to a cohomology long exact sequence

$$H^n(A) \xrightarrow{\delta} H^n(B) \xrightarrow{i} H^n(\mathrm{cone}(f)) \xrightarrow{p} H^{n+1}(A) \xrightarrow{\delta} H^{n+1}(B).$$

Proposition 3.1.1. *The boundary map $\delta\colon H^{n-1}(A[1]) \to H^n(B)$ in the long exact sequence above is equal to the map $H^n(A) \to H^n(B)$ induced by f.*

Proof. For the moment we assume that the objects of $\mathcal{A}$ have underlying sets. Let $a \in H^n(A)$ be an element. It is the image of $(a, 0) \in H^{n-1}(\mathrm{cone}(f)_{n-1})$ under the projection $H^{n-1}(\mathrm{cone}(f)) \to H^{n-1}(A[1])$. Applying the differential to this element we obtain $(-da, fa) = (0, fa)$, since a is a cocycle. Clearly, this element comes from $fa \in H^{n-1}(B)$. The case of arbitrary $\mathcal{A}$ is done by applying Freyd-Mitchell embedding theorem. □

So, cones can be used to associate a long exact cohomology sequence to arbitrary morphism of complexes.

Proposition 3.1.2. *The morphism $A \xrightarrow{f} B$ is a quasi-isomorphism if and only if* cone(f) *is acyclic.*

Proof. Follows at once from the long exact sequence and proposition 3.1.1. □

3.2 Hom complex

Let R be a commutative ring, and $\mathcal{A}$ an abelian category over R. Let A, B be complexes of objects of $\mathcal{A}$. Denote $\mathrm{Hom}^n(A, B)$ the R-module of maps $f\colon A \to B[n]$ which do not necessarily commute with differentials. In other words,

$$\mathrm{Hom}^n(A, B) = \prod_{i \in \mathbf{Z}} \mathrm{Hom}(A^i, B^{i+n}).$$

If $f \in \mathrm{Hom}^n(A, B)$ then $f^i\colon A^i \to B^{i+n}$ will denote the i-th coordinate.

A differential of an element $f \in \mathrm{Hom}^n(A,B)$ is defined as follows:

$$(d^n_{(A,B)} f)^i = d_B^{i+n} \circ f^i - (-1)^n f^{i+1} \circ d_A^i.$$

Since $d^{n+1}_{(A,B)} \circ d^n_{(A,B)} = 0$ for each n, we obtain a complex $\mathrm{Hom}^\bullet(A,B)$.

Given three complexes A, B, C there are composition maps

$$\mathrm{Hom}^m(B,C) \otimes_{\mathbf{Z}} \mathrm{Hom}^n(A,B) \xrightarrow{\circ} \mathrm{Hom}^{n+m}(A,C).$$

Hence an element $f \in \mathrm{Hom}^m(B,C)$ defines a map

$$f_*\colon \mathrm{Hom}^\bullet(A,B) \to \mathrm{Hom}^\bullet(A,C)[m],$$
$$(f_*)^n \colon u \mapsto f \circ u.$$

Since f_* is an element of $\mathrm{Hom}^n(\mathrm{Hom}^\bullet(A,B), \mathrm{Hom}^\bullet(A,C))$, it makes sense to ask what is the differential of this element. The answer is:

$$d(f_*) = (df)_*.$$

Similarly, an element $f \in \mathrm{Hom}^n(A,B)$ defines a map

$$f^*\colon \mathrm{Hom}^\bullet(B,C) \to \mathrm{Hom}^\bullet(A,C)[m],$$
$$(f^*)^n \colon u \mapsto (-1)^{nm} u \circ f.$$

Notice the sign change. With such a sign convention we obtain a formula

$$d(f^*) = (df)^*.$$

Proposition 3.2.1. $\mathrm{Hom}^\bullet$ *is a functor* $\mathrm{Comp}(\mathcal{A})^\circ \times \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(R)$. *It descends to the level of homotopy categories.*

Proof. Let $g\colon B \to C$ be a morphism of complexes. The map $g_*\colon \mathrm{Hom}^\bullet(A,B) \to \mathrm{Hom}^\bullet(A,C)$ is a morphism of complexes since $d(g_*) = (dg)_* = 0$. If $g\colon B \to C[-1]$ is a homotopy, then $(dg)_* = d(g_*)$ is a homotopy. Similar reasoning works for the contravariant argument of $\mathrm{Hom}^\bullet$. ☐

Throughout the rest of this section we will view $\mathrm{Hom}^\bullet$ as a functor from $\mathrm{Comp}(\mathcal{A})^\circ \times \mathrm{Comp}(\mathcal{A})$ to $\mathrm{Comp}(R)$.

Proposition 3.2.2. $H^0(\mathrm{Hom}^\bullet(A,B)) = \mathrm{Hom}_{K(\mathcal{A})}(A,B)$.

Proof. Omitted. ☐

Proposition 3.2.3. $\mathrm{Hom}^\bullet(A, B[1]) = \mathrm{Hom}^\bullet(A,B)[1]$.

Proof. Omitted. ☐

Proposition 3.2.4. *Let T be a complex, and $f\colon A \to B$ a morphism of complexes.* $\mathrm{Hom}^\bullet(T,-)$ *transforms the short exact sequence associated to the cone of f to the short exact sequence associated to the cone of* $f_*\colon \mathrm{Hom}^\bullet(T,A) \to \mathrm{Hom}^\bullet(T,B)$.

Proof. Omitted. □

The behaviour of $\mathrm{Hom}^\bullet$ with respect to the first argument is more complicated. One needs natural transformations which change signs in a nontrivial way.

Let us define a map $\alpha_1\colon \mathrm{Hom}^\bullet(A[1],B) \to \mathrm{Hom}^\bullet(A,B)[-1]$. Notice that

$$\mathrm{Hom}^n(A[1],B) = \mathrm{Hom}^{n-1}(A,B) = (\mathrm{Hom}^\bullet(A,B)[-1])^n.$$

The map α_1^n acts on $\mathrm{Hom}^{n-1}(A,B)$ by multiplication by $(-1)^n$.

Proposition 3.2.5. $\alpha_1\colon \mathrm{Hom}^\bullet(A[1],B) \to \mathrm{Hom}^\bullet(A,B)[-1]$ *is a natural isomorphism of complexes.*

Proof. Let $u \in \mathrm{Hom}^{n-1}(A,B)$ be an element. The differential of $\mathrm{Hom}^\bullet(A[1],B)$ transforms it to $d^{n-1}_{(A,B)}u$, while the differential of $\mathrm{Hom}^\bullet(A,B)[-1]$ sends u to $-d^{n-1}_{(A,B)}u$. Hence $\alpha_1(d^n_{(A[1],B)}u) = d^n_{(A,B)[-1]}\alpha_1(u)$. □

Let T be a complex, and $f\colon A \to B$ be a morphism of complexes. We want to define a map $\beta\colon \mathrm{Hom}^\bullet(\mathrm{cone}(f),T) \to \mathrm{cone}(f^*[-1])$. By construction,

$$\mathrm{Hom}^n(\mathrm{cone}(f),T) = \mathrm{Hom}^n(B,T) \oplus \mathrm{Hom}^{n-1}(A,T) = \mathrm{cone}(f^*[-1])^n.$$

We let β^n act on $\mathrm{Hom}^n(B,T) \oplus \mathrm{Hom}^{n-1}(A,T)$ by sending an element (u,v) to $(u,(-1)^n v)$.

Proposition 3.2.6. $\beta\colon \mathrm{Hom}^\bullet(\mathrm{cone}(f),T) \to \mathrm{cone}(f^*[-1])$ *is an isomorphism of complexes.*

Proof. A direct computation shows that the differential of $\mathrm{Hom}^\bullet(\mathrm{cone}(f),B)$ sends an element $(u,v) \in \mathrm{Hom}^n(B,T) \oplus \mathrm{Hom}^{n-1}(A,T)$ to

$$(d^n_{(B,T)}u, d^{n-1}_{(A,T)} - (-1)^n f^* u),$$

while the differential of $\mathrm{cone}(f^*[-1])$ sends it to

$$(d^n_{(B,T)}u, -d^{n-1}_{(A,T)}v + f^* u).$$

Now it is a formality to check that β commutes with the differentials. □

Proposition 3.2.7. *Let $f\colon A \to B$ be a morphism of complexes. Consider a short exact sequence associated to the cone of f:*

$$0 \longrightarrow B \xrightarrow{\ i\ } \mathrm{cone}(f) \xrightarrow{\ p\ } A[1] \longrightarrow 0.$$

Let T be a complex. There is a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}^\bullet(A[1],T) & \xrightarrow{p^*} & \mathrm{Hom}^\bullet(\mathrm{cone}(f),T) & \xrightarrow{i^*} & \mathrm{Hom}^\bullet(B,T) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta} & & \| \\ \mathrm{Hom}^\bullet(A,T)[-1] & \longrightarrow & \mathrm{cone}(f^*[-1]) & \longrightarrow & \mathrm{Hom}^\bullet(B,T). \end{array}$$

The lower row of this diagram is the short exact sequence associated to the cone of $f^*[-1]\colon \mathrm{Hom}^\bullet(B,T)[-1] \to \mathrm{Hom}^\bullet(A,T)[-1]$.

Proof. It is a straightforward computation. □

3.3 Triangles

Let $\mathcal{A}$ be an abelian category.

Definition 3.3.1. A triangle in $K(\mathcal{A})$ is a diagram of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

A triangle is called distinguished, if it is isomorphic to a triangle of the form

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{p} A[1],$$

where the second and the third map are as in (13).

In the category $K(\mathcal{A})$ distinguished triangles play the role of short exact sequences.

Proposition 3.3.2. *Let T be a complex. Let $A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{p} A[1]$ be a distinguished triangle. It induces long exact sequences*

$$\begin{array}{ccccc} \mathrm{Hom}(T,A[n]) & \xrightarrow{f[n]_*} & \mathrm{Hom}(T,B[n]) & \xrightarrow{i[n]_*} & \mathrm{Hom}(T,C[n]) \\ & & & & \downarrow{\scriptstyle p[n]_*} \\ & & & & \mathrm{Hom}(T,A[n+1]) \longrightarrow \end{array}$$

and

$$\begin{array}{ccccc} \mathrm{Hom}(C[n],T) & \xrightarrow{i[n]^*} & \mathrm{Hom}(B[n],T) & \xrightarrow{f[n]^*} & \mathrm{Hom}(A[n],T) \\ & & & & \downarrow{\scriptstyle p[n]^*} \\ & & & & \mathrm{Hom}(C[n-1],T) \longrightarrow \end{array}$$

of Hom*-modules in $K(\mathcal{A})$.*

We stress that a similar claim is completely false for $\mathrm{Comp}(\mathcal{A})$ in place of $K(\mathcal{A})$.

Proposition 3.3.3. *For every $A \in K(\mathcal{A})$ the cone of $\mathrm{id}_A \colon A \to A$ is zero.*

Proof. Let $C = \mathrm{cone}(\mathrm{id}_A)$. According to the long exact sequences above, $\mathrm{Hom}_{K(\mathcal{A})}(C, C) = 0$. So, $\mathrm{id}_C = 0$. □

Proposition 3.3.4. *Let $A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{p} A[1]$ be a triangle. If it is distinguished, then $i \circ f = 0$, $p \circ i = 0$, and $f[1] \circ p = 0$ in $K(\mathcal{A})$.*

Proof. Follows from the long exact sequences above. □

Notice that the fact that $i \circ f = 0$ and $f[1] \circ p = 0$ is not instantly obvious. One needs homotopies to trivialize these compositions.

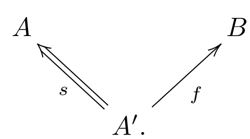
4 The derived category

4.1 Verifying Ore conditions

Theorem 4.1.1. *Let $\mathcal{A}$ be an abelian category. Consider the homotopy category $K(\mathcal{A})$. The collection of quasi-isomorphisms in $K(\mathcal{A})$ satisfies left, and right Ore conditions.*

Proof. An isomorphism is clearly a quasi-isomorphism, and 2-out-of-3 property is also immediate. For the rest we restrict ourselves to left Ore conditions.

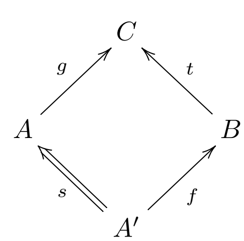
Let us verify the second condition. Let A, B be complexes. Consider a right fraction



Let C be the cone of (s, f). By definition, it fits into a a distinguished triangle

$$A' \xrightarrow{(s,f)} A \oplus B \xrightarrow{i} C \xrightarrow{p} A'[1].$$

Let $j_A \colon A \to A \oplus B$, $j_B \colon B \to A \oplus B$ be the inclusions. Set $g = i j_A s$, $t = -i j_A f$. The fact that $i \circ (s, f) = 0$ means that the diagram



commutes. We want to show that t is a quasi-isomorphism.

Consider long exact cohomology sequence of C:

$$H^n(A') \xrightarrow{H^n(s,f)} H^n(A) \oplus H^n(B) \xrightarrow{H^n(i)} H^n(C) \xrightarrow{H^n(p)} H^{n+1}(A').$$

The composition of $H^n(s,f)$ with the projection $H^n(A) \oplus H^n(B) \to H^n(A)$ is $H^n(s)$, which is an isomorphism. Hence $H^n(s,f)$ is split, and therefore injective. Since it holds for all n, $H^n(p) = 0$, and $H^n(i)$ is surjective. We thus obtain short exact sequences

$$0 \to H^n(A') \xrightarrow{H^n(s,f)} H^n(A) \oplus H^n(B) \xrightarrow{H^n(i)} H^n(C) \to 0.$$

As observed above, this sequence splits on the left, and the kernel of the splitting map is exactly $0 \oplus H^n(B)$. Hence $H^n(t) = H^n(i)H^n(j_B)$ is an isomorphism.

Let us verify the third condition. Due to additivity it is enough to prove the following. If $f\colon A \to B$ is such a map, that there exists a quasi-isomorphism $s\colon A' \to A$ with the property that $fs = 0$, then there exists a quasi-isomorphism $t\colon B \to B'$ with the property that $tf = 0$.

Let C be the cone of s, and let $i\colon A \to C$ be the inclusion map. Since $fs = 0$, the long exact sequence of Hom-modules shows that there exists $g\colon C \to B$ such that $gi = f$. Let B' be the cone of g, and $t\colon B \to B'$ the inclusion map. By construction, $tf = tgi = 0$. On the other hand, C is acyclic because s is a quasi-isomorphism. So, the long exact sequence of $B' = \mathrm{cone}(g)$ shows that t is a quasi-isomorphism. □

On the contrary the collection of quasi-isomorphisms in $\mathrm{Comp}(\mathcal{A})$ satisfies neither left nor right Ore conditions in general.

Corollary 4.1.2. *Let $\mathcal{A}$ be an abelian category equivalent to a small category. The derived categories $D(\mathcal{A})$, $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ exist.*

Proof. Indeed, $K(\mathcal{A})$ is equivalent to a small category. After replacing it by a small category we see that the collection of quasi-isomorphisms in it becomes a set. Hence this collection is locally small both on the left an on the right. Since it satisfies left, and right Ore conditions, we obtain the conclusion. □

This corollary applies, for example, to the category of modules of finite type over a ring,[2] or coherent $\mathcal{O}_X$-modules over a noetherian scheme.

[2]Stictly speaking, the fact that a naively defined category of modules of finite type over a ring is equivalent to a small category requires a choice over a proper class.

4.2 K-injective, and K-projective complexes

Let $\mathcal{A}$ be an abelian category.

Definition 4.2.1. A complex $A \in K(\mathcal{A})$ is called acyclic if $H^n(A) = 0$ for all n.

Definition 4.2.2. A complex $I \in K(\mathcal{A})$ is called K-injective if

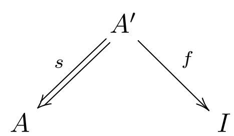
$$\mathrm{Hom}_{K(\mathcal{A})}(A, I) = 0$$

for every acyclic complex A. A complex P is called K-projective if

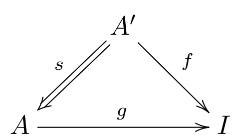
$$\mathrm{Hom}_{K(\mathcal{A})}(P, A) = 0$$

for every acyclic complex A.

Proposition 4.2.3. *If a complex I is K-injective, and A is arbitrary, then every right fraction in $K^*(\mathcal{A})$*



can be completed to a commutative triangle



in a unique way. A dual statement holds for K-projective complexes and left fractions.

Proof. Let $C = \mathrm{cone}(s)$. Since s is a quasi-isomorphism, C is acyclic, and so is $C[-1]$. Thus $\mathrm{Hom}_{K(\mathcal{A})}(C, I) = 0$, and $\mathrm{Hom}_{K(\mathcal{A})}(C[-1], I) = 0$. Applying $\mathrm{Hom}_{K(\mathcal{A})}(-, I)$ to the triangle $A \xrightarrow{s} A' \to C \to A[1]$ we conclude that $\mathrm{Hom}_{K(\mathcal{A})}(-, I)$ transforms s to an isomorphism, which is precisely the claim of the proposition. □

Proposition 4.2.4. *A bounded below complex of injective objects is K-injective. A bounded above complex of projective objects is K-projective.*

Proof. Let I be a bounded below complex of injective objects, and A an acyclic complex. Let $f\colon A \to I$ be a morphism of complexes. We are going to construct a map $h\colon A \to I[-1]$ such that $f = d_I \circ h + h \circ d_A$.

Suppose that there are morphisms $h^n\colon A^n \to I^{n-1}$, $h^{n-1}\colon A^{n-1} \to I^{n-2}$, such that $f^{n-1} = d_I^{n-2} \circ h^{n-1} + h^n \circ d_A^{n-1}$. By assumptions,

$$d_I^{n-1} \circ h^n \circ d_A^{n-1} = d_I^{n-1} \circ (f^{n-1} - d_I^{n-2} \circ h^{n-1}) = d_I^{n-1} \circ f^{n-1}.$$

Hence

$$(f^n - d_I^{n-1} \circ h^n) \circ d_A^{n-1} = f^n \circ d_A^{n-1} - d_I^{n-1} \circ f^{n-1} = 0.$$

This in turn means that the map $(f^n - d_I^{n-1} \circ h^n)\colon A^n \to I^n$ factors through $d_A^{n-1}(A^{n-1})$. Since A is acyclic, the map $d_A^n\colon A^n/d_A^{n-1}(A^{n-1}) \to A^{n+1}$ is an injection. Since I^n is injective, there exists a map $h^{n+1}\colon A^{n+1} \to I^n$, such that

$$d_A^n \circ h^{n+1} = f^n - d_I^{n-1} \circ h^n.$$

In other words, h^{n+1} continues the homotopy.

To start the homotopy, take an N such that $I^n = 0$ for all $n \leqslant N-1$. Set $h^n = 0$ for $n \leqslant N$. Since $f^n = 0$ as soon as $n \leqslant N-1$, the condition $f^n = d_{n-1}^I \circ h^n + h^{n+1} \circ d_A^n$ is satisfied for $n \leqslant N-1$. Thus we can continue to construct the homotopy from the step $N+1$, as described above.

The case of projective objects is dealt with in a similar way. □

Theorem 4.2.5. *Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ has enough injectives.*

(1) *The collection of quasi-isomorphisms in $K^+(\mathcal{A})$ is locally small on the left. As a consequence, $D^+(\mathcal{A})$ exists, and can be constructed as in theorem 2.0.14 (2).*

(2) *Let $I \in K^+(\mathcal{A})$ be a K-injective complex, and $A \in K^+(\mathcal{A})$ an arbitrary complex. The natural map*

$$\operatorname{Hom}_{K^+(\mathcal{A})}(A, I) \to \operatorname{Hom}_{D^+(\mathcal{A})}(A, I)$$

is an isomorphism.

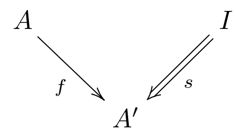
(3) *Let $K^+(\operatorname{Inj}\mathcal{A}) \subset K^+(\mathcal{A})$ be the full subcategory of K-injective complexes. The quotient functor $Q\colon K^+(\mathcal{A}) \to D^+(\mathcal{A})$ induces an equivalence of $K^+(\operatorname{Inj}\mathcal{A})$ and $D^+(\mathcal{A})$.*

An analogous statement with $+$ replaced by $-$ holds if $\mathcal{A}$ is an abelian category having enough projectives.

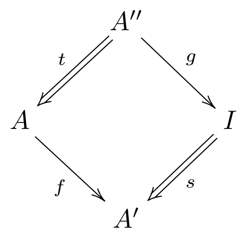
Proof. (1) Since $\mathcal{A}$ has enough injectives, for every complex $A \in K^+(\mathcal{A})$ there exists a quasi-isomorphism $s\colon A \to I$, where $I \in K^+(\mathcal{A})$ is a complex of injective objects. Thus I is K-injective. Proposition 4.2.3 shows that every quasi-isomorphism $t\colon A \to A'$ factors through $s\colon A \to I$. Thus in the definition 2.0.11 (2) we can take $S_A = \{s\colon A \to I\}$.

(2) We will repeatedly use the fact that the collection of quasi-isomorphisms satisfies both left and right Ore conditions.

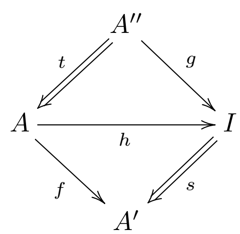
Consider a left fraction



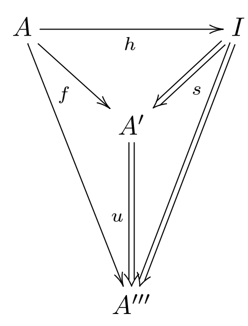
and complete it to a square



By proposition 4.2.3 there exists a morphism $h\colon A \to I$ such that the upper triangle in the diagram

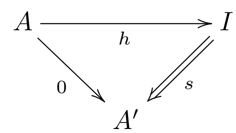


is commutative. Hence $sht = sg = ft$. As a consequence, there exists a quasi-isomorphism $u\colon A' \to A'''$ such that $ush = uf$. Next, consider the diagram

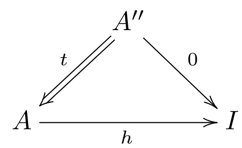


The outer fraction in this diagram dominates both the initial fraction, and the fraction $Q(h)$. Therefore the natural map $\mathrm{Hom}_{K^+(\mathcal{A})}(A, I) \to \mathrm{Hom}_{D^+(\mathcal{A})}(A, I)$ is surjective.

Proving that this map is injective instantly reduces to the following problem. Consider a commutative triangle



We want to show that $h = 0$. Since $sh = 0$ there exists a quasi-isomorphism $t\colon A'' \to A$ such that the diagram



is commutative. Unicity part of proposition 4.2.3 shows that $h = 0$.

(3) By (2) the quotient functor $Q\colon K^+(\mathcal{A}) \to D^+(\mathcal{A})$ restricted to $K^+(\operatorname{Inj}\mathcal{A})$ is fully faithful. Since every complex in $K^+(\mathcal{A})$ has a bounded below injective resolution, the functor $Q\colon K^+(\operatorname{Inj}\mathcal{A}) \to D^+(\mathcal{A})$ is essentially surjective. Therefore it is an equivalence of categories. □

Proposition 4.2.6. *Let $\mathcal{A}$ be an abelian category having enough injectives. Let A, B be objects of $\mathcal{A}$. Consider them as complexes with objects placed in degree 0. There is a natural isomorphism*

$$\operatorname{Hom}_{D^+(\mathcal{A})}(A, B[n]) \to \operatorname{Ext}^n_{\mathcal{A}}(A, B).$$

In particular, $\mathcal{A}$ is a full subcategory of $D^+(\mathcal{A})$.

In fact this proposition holds without the assumption that $\mathcal{A}$ has enough injectives.

Proof. Replacing B by its injective resolution I we obtain an isomorphism $\operatorname{Hom}_{D^+(\mathcal{A})}(A, B[n]) \to \operatorname{Hom}_{D^+(\mathcal{A})}(A, I[n])$. Since $I[n]$ is K-injective, the latter Hom-module is naturally isomorphic to $\operatorname{Hom}_{K^+(\mathcal{A})}(A, I[n]) = H^n(\operatorname{Hom}^\bullet(A, I))$. But $\operatorname{Hom}^\bullet(A, I)$ is just a result of termwise application of $\operatorname{Hom}_{\mathcal{A}}(A, -)$ to I, the resolution of B. Hence $H^n(\operatorname{Hom}^\bullet(A, I)) = R^n \operatorname{Hom}_{\mathcal{A}}(A, B) = \operatorname{Ext}^n_{\mathcal{A}}(A, B)$. □

4.3 Constructing derived functors

Proposition 4.3.1. *Let $\mathcal{A}$ be an abelian category having enough injectives, and $i\colon K^+(\operatorname{Inj}\mathcal{A}) \to K^+(\mathcal{A})$ the inclusion functor. There exists a functor*

$$r\colon K^+(\mathcal{A}) \to K^+(\operatorname{Inj}\mathcal{A}),$$

and a natural transformation $\tau\colon \operatorname{id}_{\mathcal{A}} \to ir$ such that r sends quasi-isomorphisms to isomorphisms, and τ is a quasi-isomorphism at each object.

Proof. For every object $A \in K^+(\mathcal{A})$ pick an injective resolution $\tau_A \colon A \to I_A$,[3] and set $r(A) = I_A$.

Let $f \colon A \to B$ be a morphism. Consider a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow{\scriptstyle \tau_A} & & \Big\downarrow{\scriptstyle \tau_B} \\ I_A & & I_B. \end{array}$$

Since τ_A is a quasi-isomorphism, there exists a unique morphism $I_f \colon I_A \to I_B$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow{\scriptstyle \tau_A} & & \Big\downarrow{\scriptstyle \tau_B} \\ I_A & \xrightarrow{I_f} & I_B \end{array}$$

is commutative. Set $r(f) = I_f$. By unicity $r(\mathrm{id}_A) = \mathrm{id}_A$, and $r(gf) = r(g)r(f)$, so r is a functor. Naturality of $\tau \colon \mathrm{id}_{\mathcal{A}} \to i \circ r$ is clear. $\square$

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Assume that $\mathcal{A}$ has enough injectives, and that $D^+(\mathcal{B})$ exists. Let $Q_{\mathcal{A}} \colon K^+(\mathcal{A}) \to D^+(\mathcal{A})$, and $Q_{\mathcal{B}} \colon K^+(\mathcal{B}) \to D^+(\mathcal{B})$ be quotient functors.

Pick a functor $r \colon K^+(\mathcal{A}) \to K^+(\operatorname{Inj} \mathcal{A})$ as in proposition 4.3.1. Since r sends quasi-isomorphisms to isomorphisms, it gives rise to a functor $\widehat{r} \colon D^+(\mathcal{A}) \to K^+(\operatorname{Inj} \mathcal{A})$, and a natural isomorphism $\eta \colon r \to \widehat{r} Q_{\mathcal{A}}$.

Let $F \colon \mathcal{A} \to \mathcal{B}$ a functor. Consider a functor $F \colon K^+(\mathcal{A}) \to K^+(\mathcal{B})$ obtained by termwise application of F. Define a functor $RF \colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ as

$$RF = Q_{\mathcal{B}} F i \widehat{r},$$

where $i \colon K^+(\operatorname{Inj} \mathcal{A}) \to K^+(\mathcal{A})$ is the inclusion functor.

We also define a natural transformation $\mu \colon Q_{\mathcal{B}} F \to RF \circ Q_{\mathcal{A}}$ as

$$\mu = Q_{\mathcal{B}} F(i(\eta)\tau).$$

The functor RF is called the total right derived functor of F. Notice that for an object $A \in \mathcal{A}$ viewed as a complex

$$H^n(RF(Q_{\mathcal{A}} A)) \cong H^n(Fir(A)) = H^n(F(I_A)) = R^n F(A),$$

where $R^n F$ is the usual n-th right derived functor of F. Also notice that to define RF we do not need to assume that F is left exact (and even additive). If F is not left exact then it is not true in general that $H^0(RF(A)) = F(A)$.

[3] Since the collection of objects of $\mathcal{A}$ is not a set in general, this step relies on a sort of axiom of choice over proper classes.

Proposition 4.3.2. *The pair* (RF, μ) *has the following universal property. Given a functor* $G\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$*, and a natural transformation* $\mu'\colon Q_{\mathcal{B}}F \to GQ_{\mathcal{A}}$ *there exists a unique natural transformation* $\xi\colon RF \to G$ *such that* $\xi_{Q_{\mathcal{A}}}\mu = \mu'$.

Proof. Since $\tau\colon \mathrm{id}_{K^+(\mathcal{A})} \to ir$ is a quasi-isomorphism, $Q_{\mathcal{A}}(\tau)$ is an isomorphism, so $GQ_{\mathcal{A}}(\tau)\colon GQ_{\mathcal{A}} \to GQ_{\mathcal{A}}ir$ is an isomorphism. Hence there exists a unique natural transformation $\psi\colon Q_{\mathcal{B}}Fir \to GQ_{\mathcal{A}}$ such that the diagram

$$\begin{array}{ccc} Q_{\mathcal{B}}F & \xrightarrow{\mu'} & GQ_{\mathcal{A}} \\ {\scriptstyle Q_{\mathcal{B}}F(\tau)}\downarrow & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle GQ_{\mathcal{A}}(\tau)} \\ Q_{\mathcal{B}}Fir & \xrightarrow{\mu'_{ir}} & GQ_{\mathcal{A}}ir. \end{array}$$

is commutative.

The functor $Q_{\mathcal{A}}$ is surjective on objects, by construction. Hence we can define a natural transformation $\xi\colon RF \to G$ by associating the arrow $\psi_A \circ i(\eta)^{-1}$ with an object A. One then easily checks that ξ is indeed a natural transformation. Unicity of ξ follows from the fact that ψ is unique. □

Thus the functor RF constructed in such a way is in some sense unique. In particular, a different choice of functorial injective resolutions (r, τ) gives rise to an isomorphic total right derived functor RF.

There is a general notion of total right (left) derived functors on derived or triangulated categories (see [1], [4] for details). Since it is based on a notion of an exact functor between triangulated categories, we refrain from discussing it.

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