Vanishing of proper higher direct images

Nicolas Müller

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All references of the form [Tag ****] are to the Stacks Project [Stacks]. The aim of these notes is to prove the following

Theorem 1 ([Tag 095U])**.** *Let $f : X \to S$ be a proper morphism of noetherian schemes with fibers of dimension $\leqslant d$ and let $\mathcal{F}$ be a torsion sheaf on $X_{\text{ét}}$. Then $R^q f_* \mathcal{F} = 0$ for $q > 2d$.*

1 Reductions

Lemma 2. *It suffices to show Theorem 1 for the case when S is the spectrum of an algebraically closed field.*

Proof. Let $\bar{s}$ be a geometric point of S. We have the following cartesian diagram:

$$\begin{array}{ccc} X_{\bar{s}} & \xrightarrow{\bar{s}'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \bar{s} & \xrightarrow{\bar{s}} & S \end{array} .$$

By the proper base change theorem [Tag 095T] we have

$$(R^q f_* \mathcal{F})_{\bar{s}} = (\bar{s}^{-1} R^q f_* \mathcal{F})_{\bar{s}} \cong (R^q f'_* \bar{s}'^{-1} \mathcal{F})_{\bar{s}}.$$

Therefore, if Theorem 1 is true for schemes proper over $\bar{s}$ it is true for schemes proper over S. □

From now on, we assume that $S = \operatorname{Spec} k$, where k is an algebraically closed field and $\dim X = d$. By the topological invariance of the étale site [Tag 03SI], we may assume that X is reduced.

Remark By [Tag 03Q9], we have $(R^q f_* \mathcal{F})_{\operatorname{Spec} \bar{k}} = H^q_{\text{ét}}(X, \mathcal{F})$. Hence, Theorem 1 is equivalent to $H^q_{\text{ét}}(X, \mathcal{F}) = 0$ for all $q > 2d$.

Lemma 3 (Domination Trick)**.** *Let X' be a scheme of dimension d. Let $\pi : X' \to X$ be a proper morphism such that there is a dense open subscheme $U \subseteq X$ such that $\pi_U : X' \times_X U \to U$ is an isomorphism and* $\operatorname{codim}_X X \smallsetminus U \geqslant m$ *and the fibers of π have dimension $< m$. Suppose that Theorem 1 is true for*

1. dimensions $< d$ and

2. for $f \circ \pi$.

Then it is true for f.

Proof. Note that π is surjective, since it is closed and its image contains a dense subset. Write $\mathcal{G} := \pi^{-1}\mathcal{F}$. Then the natural map $\mathcal{F} \to \pi_*\mathcal{G}$ is injective and we obtain a short exact sequence

$$0 \to \mathcal{F} \to \pi_*\mathcal{G} \to \mathcal{Q} \to 0$$

for some $\mathcal{Q} \in Ab(X_{\text{ét}})$ with $\mathcal{Q}|_U = 0$. Let $Y := X \setminus U$ and let $i : Y \to X$ denote the closed embedding. By [Tag 04CA] we have $\mathcal{Q} \cong i_* i^{-1}\mathcal{Q}$. Because we assume Theorem 1 is true for dimension $< d$ and $\operatorname{codim}_X Y > 0$ we have $H^p(Y, i^{-1}\mathcal{Q}) = 0$ for $p > 2(d-m)$. Since i is finite, the Leray spectral sequence [Tag 0733] yields (see also Noah's talk from last semester) $H^p(X, \mathcal{Q}) = H^p(Y, i^{-1}\mathcal{Q}) = 0$ for $p > 2(d-m)$. Hence the above short exact sequence yields the following long exact sequence

$$\cdots \to 0 \to H^p(X, \mathcal{F}) \to H^p(X, \pi_*\mathcal{G}) \to 0 \to \dots$$

for all $p > 2d$. Hence, it suffices to prove $H^p(X, \pi_*\mathcal{G}) = 0$ for $p > 2d$. To prove this, we will use the Leray spectral sequence [Tag 0732]. Its object on the second page are $E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{G})$ the differentials are of bidegree $(r, -r+1)$ and it converges to $H^{p+q}(X', \mathcal{G})$.

By assumption we have $R^q\pi_*\mathcal{G} = 0$ for $q > 2(m-1)$ and hence

$$E_r^{p,q} = E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{G}) = 0 \text{ for } q > 2(m-1) \text{ and } r \geqslant 2 \text{ and any } p. \tag{1}$$

Furthermore, for $q > 0$ we have $(R^q\pi_*\mathcal{G})|_U = 0$ since π is an isomorphism over U. Hence, as above, we obtain

$$E_r^{p,q} = E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{G}) = 0 \text{ for } q > 0 \text{ and } p > 2(d-m) \text{ and } r \geqslant 2. \tag{2}$$

Suppose from now on that $p > 2d$. We want to show that the $E_r^{p,0} = E_{r+1}^{p,0} = \dots$ in the Leray spectral sequence for $r \geqslant 2$. The r-th differential gives the following complex

$$\cdots \to E_r^{p-r,r-1} \to E_r^{p,0} \to E_r^{p+r,1-r} = 0 \to \dots$$

The right-hand side is 0 as $r > 1$. By (1), the left-hand side is 0 if $r \geqslant 2m$. If $r < 2m$, then $r-1 > 0$ and $p-r > p-2m > 2(d-m)$ and the left-hand side is 0 by (2). Therefore $E_r^{p,0}$ stabilizes from $r = 2$ and therefore $H^p(X, \pi_*\mathcal{G}) = H^p(X', \mathcal{G}) = 0$ by assumption. □

Lemma 4 (Composition Trick)**.** *Let $f : X \to Y$ and $g : Y \to \operatorname{Spec} k$ be proper morphisms such that $\dim X_y \leqslant \dim X - \dim Y$ for all $y \in Y$. If Theorem 1 holds for f and g, then it holds for $g \circ f$.*

Proof. Let $\mathcal{F} \in Ab(X_{\text{ét}})$ be a torsion sheaf, let $d := \dim X$ and $e := \dim Y$. We again use the Leray spectral sequence [Tag 0732]. We have $E_2^{p,q} = H^p(Y, R^q\pi_*\mathcal{F})$. If $p > 2e$ or $q > 2(d-e)$, then $H^p(Y, R^q\pi_*\mathcal{F}) = 0$, by assumption. Hence $E_r^{p,q} = 0$ for all $r \geqslant 2$ in this case. As the sequence converges to $H^{p+q}(X, \mathcal{F})$ we obtain $H^n(X, \mathcal{F}) = 0$ for $n > 2d$, as desired. □

Lemma 5. *It suffices to prove Theorem 1 for the case when X is integral normal proper over an algebraically closed field.*

Proof. We proceed by induction on $d = \dim X$. The base case $d = 0$ is the case of finite morphisms [Tag 03QP]. Suppose that $d > 0$. Let $\pi : X' \to X$ be the normalization of X. Since X is reduced, there is an open dense subset $U \subseteq X$ such that $\pi_U : X' \times_X U \to U$ is an isomorphism ([Tag 0BXR], [Tag 0BAC]). By [Tag 0BXR], the morphism π is finite, hence proper. The normalization X' is the disjoint union of integral normal schemes. By Lemma 3 it suffices to prove the Theorem for $f' := f \circ \pi$. Let $\mathcal{G} := \pi^{-1}\mathcal{F}$. Let $j_i : U_i \to X'$ denote the connected components of X'. Note that $\mathcal{G} \cong \bigoplus_{i=1}^n j_{i*}j^{-1}\mathcal{G}$. By the Leray spectral sequence [Tag 01F4] we have $H^q(X', j_{i*}j^{-1}\mathcal{G}) \cong H^q(U_i, j^{-1}\mathcal{G})$ and hence we may assume without loss of generality that X' is connected. □

Suppose from now on that X is integral normal proper over an algebraically closed field k.

Lemma 6. *It suffices to prove the theorem for $X = \mathbb{P}^1_k$.*

Proof. By the previous lemmas we may assume that X is integral normal proper over an algebraically closed field. For $d = 0$, this is the case of finite morphisms, proved in Emil's talk ([Tag 03QP]). We prove the claim by induction. Suppose that $d > 0$: Choose a rational function $f : X \dashrightarrow \mathbb{A}^1_k \subset \mathbb{P}^1_k$. Let $U \subset X$ be its domain of definition. Let $X' \subseteq X \times_{\operatorname{Spec} k} \mathbb{P}^1_k$ be the graph of f, that is the closure of the graph of $f|_U$. Let $b : X' \to X$ denote the first projection and let $g : X' \to \mathbb{P}^1_k$ denote the second projection. Note that b is an isomorphism above U. Let $Y := X \smallsetminus U$. Then $\operatorname{codim}_X Y \geqslant 2$ (since normal schemes are regular of codimension 1 and hence f extends to codim 1 points by the valuative criterion of properness for $\mathbb{P}^1_k$). Since $X' \hookrightarrow X \times_{\operatorname{Spec} k} \mathbb{P}^1_k$ is a closed immersion, hence proper and $X \times_{\operatorname{Spec} k} \mathbb{P}^1_k$ is the base change of the proper morphism $X \to \operatorname{Spec} k$, the morphism g is proper. By Lemma 3 it suffices to prove Theorem 1 for X'.

Let $\mathcal{G}$ be a torsion sheaf on X'. The fibers of g have dimension $< d$. By Lemma 4 for $X' \xrightarrow{g} \mathbb{P}^1_k \to \operatorname{Spec} k$ and the induction hypothesis Theorem 1 follows for X' if it holds for $\mathbb{P}^1_k$. □

2 The étale fundamental group

To prove Theorem 1 for $\mathbb{P}^1_k$, we will need some facts about the étale fundamental group. A reference is [Tag 0BQ8] and [Tag 0BQ6]. Let X be a connected scheme and let $\bar{x}$ be a geometric point of X. Let FÉt_X denote the category of finite étale coverings of X and let $F_{\bar{x}} : \text{FÉt} \to (\text{Finite Sets})$ denote the fiber functor that maps $U \in \operatorname{Ob}(\text{FÉt})$ to the underlying set of the topological space of $U_{\bar{x}}$.

Definition The *fundamental group* of X with *base point* $\bar{x}$ is the group

$$\pi_1(X, \bar{x}) = \operatorname{Aut}(F_{\bar{x}}) = \{\text{group of natural equivalences between } F_{\bar{x}} \text{ and itself}\}$$

Remark One has the embedding

$$\pi_1(X, \bar{x}) \hookrightarrow \prod_{U \in \operatorname{Ob}(\text{FÉt})} \operatorname{Aut}(F_{\bar{x}}(U)).$$

When endowing $\mathrm{Aut}(A)$ with the discrete topology, the image of the embedding is closed and $\pi_1(X, \bar{x})$ becomes a profinite group.

Theorem 7 (see [Tag 0BND], [Tag 0DV5], [Tag 0DV6])**.**

1. *The fiber functor defines an equivalence of categories*

$$F_{\bar{x}} \colon \mathrm{F\acute{E}t}_X \to \mathrm{Finite}{-}\pi_1(X, \bar{x}){-}\mathrm{Sets}.$$

2. *Let R be a finite ring. There is an equivalence of categories*

$$(\textit{finite locally constant sheaves of } R\textit{-modules on } X_{\text{ét}}) \leftrightarrow (\textit{finite } R[\pi_1(X, \bar{x})]\textit{-modules})$$

3. *Given a morphism $f \colon Y \to X$ of connected schemes with $\bar{x} = f(\bar{y})$ there is a canonical continuous homomorphism $f_* \colon \pi_1(Y, \bar{y}) \to \pi_1(X, \bar{x})$ such that*

$$\begin{array}{ccc} \mathrm{F\acute{E}t}_X & \xrightarrow[\text{base change}]{} & \mathrm{F\acute{E}t}_Y \\ \downarrow F_{\bar{x}} & & \downarrow F_{\bar{y}} \\ \textit{finite } \pi_1(X, \bar{x})\textit{-sets} & \xrightarrow{f_*} & \textit{finite } \pi_1(Y, \bar{y})\textit{-sets} \end{array}$$

and

$$\begin{array}{ccc} \textit{f.l.c. sheaves of } R\textit{-modules on } X_{\text{ét}} & \xrightarrow{f^{-1}} & \textit{f.l.c. sheaves of } R\textit{-modules on } Y_{\text{ét}} \\ \downarrow & & \downarrow \\ \textit{finite } R[\pi_1(X, \bar{x})] \textit{ modules} & \xrightarrow{f_*} & \textit{finite } R[\pi_1(Y, \bar{y})] \textit{ modules} \end{array}$$

commute

Proposition 8 ([Tag 03SF])**.** *Let G be a finite $\pi_1(X, \bar{x})$-set of the form $G = \pi_1(X, \bar{x})/H$ for some normal open subgroup H. Then G corresponds via Theorem 7 to a connected finite étale covering $Y \to X$ such that $\mathrm{Aut}_X(Y) \cong G$. Such a Y is called* Galois cover*.*

Proposition 9 (Proposition 2.18 in [Hel18])**.** *Let Y be a Galois cover of X and let $H < \mathrm{Aut}_X(Y)$. Then $\mathrm{Aut}_{Y/H}(Y) = H$.*

Proposition 10. *Let $\rho : G \to \mathrm{Aut}(V)$ be a representation of a finite ℓ-group G on a finite-dimensional $\mathbb{F}_{\ell^n}$-vector space V. Then there is a G-stable filtration $0 = V_0 \subset \cdots \subset V_n = V$ of subspaces with $\dim V_i/V_{i-1} = 1$.*

Proof. We proceed by induction. The cases $n \in \{0, 1\}$ are clear. Suppose that $n \geqslant 2$. Then $V^G \neq \{0\}$ since otherwise $V \setminus \{0\}$ is disjoint the union of orbits of sizes of positive powers of ℓ, which is a contradiction. Let $V_1 \subset V^G$ be a 1-dimensional subspace. Then G linearly on V/V_1 and, by induction we obtain a G-stable filtration

$$0 = V_1' \subset \cdots \subset V_n' = V/V_1$$

with $\dim(V_i')/(V_{i-1}') = 1$. This lifts to a the desired filtration. $\square$

Proposition 11 ([Tag 0A3R]). *Let ℓ be a prime number and let $\mathcal{F}$ be a finite type, locally constant sheaf of $\mathbb{F}_\ell$-vector spaces on $X_{\text{ét}}$. Then, there exists a finite étale morphism $f : Y \to X$ of degree prime to ℓ such that $f^{-1}\mathcal{F}$ has a finite filtration whose successive quotients are isomorphic to $\underline{\mathbb{Z}/\ell\mathbb{Z}}$.*

Proof. By Theorem 7 the sheaf $\mathcal{F}$ corresponds to a finite $\mathbb{F}_\ell[\pi_1(X, \bar{x})]$ module V. That is, we obtain a representation $\rho\colon \pi_1(X, \bar{x}) \to \operatorname{Aut}(V)$. Let $G := \operatorname{im}\rho$. Let $H \subset G$ be an ℓ-Sylow subgroup. Let $Z \to X$ be a Galois cover with $\operatorname{Aut}_Z(X) \cong G$. Let $Y := Z/H$. Then $f\colon Y \to X$ is a connected finite étale cover. We have $\gcd(\deg f, \ell) = \gcd(|G/H|, \ell) = 1$.

Let $\bar{y} \in Y$ be a geometric point over $\bar{x}$. By Theorem 7 the action of $\pi_1(Y, \bar{y})$ on G factors through $\pi_1(X, \bar{x})$. But since $\operatorname{Aut}_Y(G) = \operatorname{Aut}_Y(Z) = H$ by Proposition 9, we obtain

$$\operatorname{im}(\pi_1(Y, \bar{y}) \to \pi_1(X, \bar{x})) \subset \rho^{-1}(H).$$

Since $\pi_1(Y, \bar{y}) \to \pi_1(X, \bar{x}) \to \operatorname{Aut}(V)$ corresponding to $f^{-1}\mathcal{F}$, we have $\operatorname{im}(\pi_1(Y, \bar{y}) \to \operatorname{Aut}(V)) \subset H$. Since H is an ℓ-group, by Proposition 10, we obtain a filtration of V with successive quotients $\cong \mathbb{Z}/\ell\mathbb{Z}$. By the equivalence of categories in Theorem 7 we obtain a filtration of $f^{-1}\mathcal{F}$ with successive quotients isomorphic to $\underline{\mathbb{Z}/\ell\mathbb{Z}}$. □

3 The trace method

Proposition 12 ([Tag 03QP]). *Let $f\colon Y \to X$ be a finite morphism. For $\mathcal{F} \in Ab(Y_{\text{ét}})$ and any geometric point $\bar{x} \in X$ we have*

$$(f_*\mathcal{F})_{\bar{x}} = \bigoplus_{\bar{y} \in Y\colon f(\bar{y}) = \bar{x}} \mathcal{F}_{\bar{y}}.$$

Proposition 13 ([Tag 03S5]). *Let $f\colon Y \to X$ be an étale morphism. For $\mathcal{F} \in Ab(Y_{\text{ét}})$ and any geometric point $\bar{x} \in X$ we have*

$$(f_!\mathcal{F})_{\bar{x}} = \bigoplus_{\bar{y} \in Y\colon f(\bar{y}) = \bar{x}} \mathcal{F}_{\bar{y}}.$$

Hence, if f is finite étale, then $f_* = f_!$. As $f_!$ is the left adjoint of f^* and f_* is its right adjoint, for $\mathcal{F} \in Ab(X_{\text{ét}})$ we obtain a map $\mathcal{F} \to f_*f^{-1}\mathcal{F} = f_!f^{-1}\mathcal{F} \to \mathcal{F}$. By [Tag 04HN], locally we have $Y = \coprod_{i=1}^n X$. Then $f_*f^{-1}\mathcal{F} = \mathcal{F}^{\oplus n}$ and for an étale neighborhood $U \to X$ we have

$$\begin{aligned} \mathcal{F}(U) &\to f_*f^{-1}\mathcal{F}(U) \to \mathcal{F}(U) \\ s &\mapsto (s, \ldots, s) \quad \mapsto n \cdot s \end{aligned}$$

This is an isomorphism, if multiplication with n is an isomorphism.

4 The case of $\mathbb{P}^1_k$

Lemma 14. *It suffices to prove Theorem 1 for the case when $\mathcal{F}$ is constructible.*

Proof. By [Tag 03SA] $\mathcal{F}$ is a filtered colimit of constructible abelian sheaves. By [Tag 03QF] taking cohomology commutes with filtered colimits. Therefore, it suffices to prove Theorem 1 for the case when $\mathcal{F}$ is constructible. $\square$

Let ℓ be a prime number.

Proposition 15. *For Y proper over $\operatorname{Spec} k$ of dimension 1, we have $H^q(Y, \underline{\mathbb{Z}/\ell\mathbb{Z}}) = 0$ for $q > 2$.*

Proof. The case when $\ell \neq \operatorname{char} k$ was proved in Sebastian's talk. Suppose that $\ell = \operatorname{char} k$. We have the Artin-Schreier sequence on $\mathbb{P}^1_k$:

$$0 \to \underline{\mathbb{Z}/\ell\mathbb{Z}} \to \mathbb{G}_a \xrightarrow{\operatorname{Frob}-1} \mathbb{G}_a \to 0.$$

Since $H^i_{\text{ét}}(Y, \mathbb{G}_a) = H^i(Y, \mathcal{O}_Y) = 0$ for $i > 1$ the long exact sequence yields the result. $\square$

Lemma 16. *Let Y be proper over $\operatorname{Spec} k$ of dimension 1. Let $j : U \to Y$ be a dense open embedding. Let $\mathcal{F} := j_! \underline{\mathbb{Z}/\ell\mathbb{Z}}$. Then $H^q(Y, \mathcal{F}) = 0$ for $q > 2$.*

Proof. Set $Z := Y \setminus U$. Consider the short exact sequence

$$0 \to j_! \underline{\mathbb{Z}/\ell\mathbb{Z}} \to \underline{\mathbb{Z}/\ell\mathbb{Z}} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is supported on Z. As before we have $H^q(Y, \mathcal{Q}) = 0$ for $q > 0$. By Proposition 15 the long exact sequence yields the result. $\square$

Theorem 17 (Zariski's Main Theorem, [Tag 05K0])**.** *Let $f : Y \to Z$ be a morphism of schemes. Assume that f is quasi-finite and separated and Z is quasi-compact and quasi-separated. Then, there exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{\quad j \quad} & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S & \end{array}$$

with j a quasi-compact open immersion and π finite.

Lemma 18. *Let $j : U \to \mathbb{P}^1_k$ be an open immersion with U non-empty and let $\mathcal{G}$ be a finite locally constant sheaf of $\mathbb{F}_\ell$-vector spaces on U. Then $H^q(\mathbb{P}^1_k, j_!\mathcal{G}) = 0$ for $q > 2$.*

Proof. Let $f : V \to U$ be finite étale morphism of degree prime to ℓ as in Proposition 11 and set $\mathcal{F} := f^{-1}\mathcal{G}$. The composition of the natural maps $\mathcal{G} \to f_*\mathcal{F} \to \mathcal{G}$ is an isomorphism. Since $j_!$ is exact it suffices to prove that $H^q(\mathbb{P}^1_k, j_! f_* \mathcal{F}) = 0$ for $q > 2$. Note that V is reduced since is finite étale over U (and hence locally just copies of opens of the reduced scheme U). By Zariski's Main Theorem 17 we obtain a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{j'} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ U & \xrightarrow{j} & \mathbb{P}^1_k \end{array}$$

with j' and open immersion and f' finite and Y reduced. By looking at the stalks and applying Propositions 12 and 13 we see that $j_! f_* \mathcal{F} = f'_* j'_! \mathcal{F}$. Since f' is finite, we have $H^q(Y, j'_! \mathcal{F}) = H^q(\mathbb{P}^1_k, f'_* j'_! \mathcal{F}) = H^q(\mathbb{P}^1_k, j_! f_* \mathcal{F})$ for $q \geqslant 0$.

By Proposition 11 we obtain a finite filtration $0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ with short exact sequences

$$0 \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \underline{\mathbb{Z}/\ell\mathbb{Z}} \to 0.$$

for $i > 0$. Since $j'_!$ is exact, we obtain short exact sequences

$$0 \to j'_! \mathcal{F}_{i-1} \to j'_! \mathcal{F}_i \to j'_! \underline{\mathbb{Z}/\ell\mathbb{Z}} \to 0.$$

By Lemma 16 we have $H^q(Y, j'_! \underline{\mathbb{Z}/\ell\mathbb{Z}}) = 0$ for $q > 2$ and hence, inductively $H^q(Y, j'_! \mathcal{F}_i) = 0$ for $q > 2$, as desired. □

Proof of Theorem 1. Without loss of generality we assume that $X = \mathbb{P}^1_k$ and $\mathcal{F}$ is constructible. By [Tag 005K] (every constructible partition of an irreducible scheme has one part which contains a dense open subset) there is an open dense subset $j : U \to \mathbb{P}^1_k$ such that $\mathcal{F}|_U$ is a finite locally constant torsion sheaf. Consider the short exact sequence

$$0 \to j_! j^{-1} \mathcal{F} \to \mathcal{F} \to \mathcal{Q} \to 0$$

for some torsion sheaves $\mathcal{Q}$ supported on $\mathbb{P}^1_k \setminus U$. As above, we have $H^q(X, \mathcal{Q}) = 0$ for $q > 0$ and hence it suffices to show that $H^q(X, j_! j^{-1} \mathcal{F}) = 0$ for $q > 2$.

We write $j^{-1} \mathcal{F} = \mathcal{F}|_U = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_r$ such that $\mathcal{F}_i$ is locally constant $\ell_i^{n_i}$-torsion for some prime ℓ_i. Since have short exact sequences

$$0 \to j_!(\mathcal{F}_i[\ell_i]) \to j_! \mathcal{F}_i \to j_!(\mathcal{F}_i / \mathcal{F}_i[\ell_i]) \to 0,$$

to show that $H^q(\mathbb{P}^1_k, j_! \mathcal{F}_i) = 0$, we may assume that $\mathcal{F}_i$ is ℓ_i-torsion. Since $j_!$ and $H^q(\mathbb{P}^1_k, -)$ commutes with direct sums, we may assume that $\mathcal{F}|_U$ is ℓ-torsion. In other words $\mathcal{F}|_U$ is a finite locally constant sheaf of $\mathbb{F}_\ell$-vector spaces on U. By Lemma 18 we have $H^q(\mathbb{P}^1_k, \mathcal{F}|_U) = 0$ for $q > 2$, as desired. □

References

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