Vanishing Theorems in Galois Cohomology

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This talk is based on The Stacks Project [3] and Serre [2], but presented in a more digestible manner suggested by Maxim.

1 Tsen's Theorem and Implications

The vanishing of $\mathrm{H}^2(\mathrm{Spec}(L), \mathbb{G}_\mathrm{m})$ for fields L is very important to study, as it implies vanishing of higher cohomology groups in certain cases.

For the proof of Tsen's theorem see Gille and Szamuely [1, Theorem 6.2.8].

Definition 1.1. A field K is called C_1 if for all integers $n > d > 0$ and every homogeneous polynomial $F \in K[X_1, \ldots, X_n]$ of degree d there exists a nontrivial zero of F in K. Equivalently, every hypersurface of degree d in $\mathbb{P}^{n-1}_K$ has a K-rational point.

Lemma 1.2. *Let K be a C_1-field. Then every algebraic extension L/K is also C_1.*

Proof. Let $f \in L[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d < n$. Choose a basis $v_1, \ldots, v_m$ of the K-vector space L. We make a change of variables by

$$x_i := \sum_{j=1}^{m} x_{ij} v_j,$$

where x_{ij} are new variables. Now consider the equation $N_{L/K}(f(x_1, \ldots, x_n)) = 0$, which becomes a homogeneous equation of degree md in mn variables over K after the change of variables. Since $md < mn$ there is a solution (α_{ij}) of this equation in K by assumption. Changing back to the initial coordinates and using the fact that the norm of an element is zero if and only if the element is zero, we find a solution of $f = 0$ in L. □

Theorem 1.3 (Tsen)**.** *Let K be a field extension of transcendence degree 1 over an algebraically closed field k. Then K is C_1.*

Proof. By Lemma 1.2 we can reduce to the purely transcendental case $K = k(t)$. Let $f \in k(t)[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d < n$. We can get rid of denominators and assume without loss of generality that $f \in k[t][x_1, \ldots, x_n]$. We choose an integer $N > 0$ and make a change of variables by

$$x_i := \sum_{j=0}^{N} a_{ij} t^j$$

for new variables a_{ij}. Plugging this into f and regrouping by powers of t we obtain an equation we need to solve:

$$0 = f(x_1, \ldots, x_n) = \sum_{i=0}^{dN+r} f_\ell(a_{10}, \ldots, a_{nN}) t^\ell$$

where r is the maximal degree of all coefficients of f and all f_ℓ are homogeneous polynomials over k in the variables a_{ij}. This equation is satisfied if and only if there exist elements $a_{ij} \in k$ such that $f_\ell(a_{10}, \ldots, a_{nN}) = 0$ for all $0 \leqslant \ell \leqslant dN + r$. So we have $dN + r + 1$ equations in $n(N + 1)$ variables, which need to have a common solution in k. For large enough N we have $dN + r + 1 \leqslant n(N + 1)$, so the equations define a nonempty Zariski closed subset of $\mathbb{P}^{nN+n-1}$, which has a k-rational point because k is algebraically closed. We conclude that f has a $k(t)$-rational point and so K is C_1. □

Corollary 1.4. *Let K be a field extension of transcendence degree 1 over an algebraically closed field k. Then* $\mathrm{Br}(K) = 0$.

Proof. Pick a separable closure K^s of K. Let D be a central division algebra over K. By Alex' talk there is a separable extension L of K which splits D. In particular, there exists an integer $n > 0$ and a homomorphism $\varphi : D \to M_n(K^s)$ which becomes an isomorphism by tensoring $\widetilde{\varphi} : D \otimes_K K^s \xrightarrow{\cong} M_n(K^s)$. Let σ be an element of $\mathrm{Gal}(K^s/K)$. By abuse of notation also denote σ for the induced endomorphism on $M_n(K^s)$ and for the induced endomorphism on $D \otimes_K K^s$. By the Noether-Skolem Theorem for the two homomorphisms $\sigma \circ \widetilde{\varphi} \circ \sigma^{-1}$ and $\widetilde{\varphi}$ there exists an invertible element $b \in M_n(K^s)$ such that $\sigma \circ \widetilde{\varphi} \circ \sigma^{-1} = b \cdot \widetilde{\varphi} \cdot b^{-1}$. For every element $d \in D \otimes_K K^s$ with $d = \sigma(d)$ we then have

$$\sigma(\det(\widetilde{\varphi}(d))) = \det(\sigma \circ \widetilde{\varphi}(d)) = \det(b \cdot \widetilde{\varphi}(d) \cdot b^{-1}) = \det(\widetilde{\varphi}(d)).$$

Therefore the determinant induces a homomorphism $\det : D \to K$. Choose a K-basis $v_1, \ldots, v_{n^2}$ of D. We then have the equation $\det(\sum_{i=1}^{n^2} x_i v_i) = 0$, which is a homogeneous polynomial over K of degree n in n^2 variables. Since D is divisible, there is no solution of this equation. By Tsen's Theorem 1.3 we conclude that therefore $n^2 > n$ and so $n = 1$, which implies $D \cong K$. □

Corollary 1.5. *Let K be a field extension of transcendence degree 1 over an algebraically closed field k. Then for every separable algebraic extension L/K we have* $\mathrm{H}^2(\mathrm{Spec}(L), \mathbb{G}_\mathrm{m}) = 0$.

Proof. Such a field L has also transcendence degree 1 over k, so by 1.4 the Brauergroup vanishes $\mathrm{Br}(L) = 0$. By Alex' talk we know that $\mathrm{H}^2(\mathrm{Spec}(L), \mathbb{G}_\mathrm{m}) \cong \mathrm{Br}(L)$ which implies the statement. □

2 Vanishing in Group Cohomology

2.1 Finite Groups

In this section let G be a finite group and $H < G$ a subgroup. By a "G-module" we mean a discrete left G-module.

Definition 2.1. For an H-module M we define G-modules

$$\operatorname{ind}_H^G M := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M, \quad g \cdot (x \otimes m) := gx \otimes m$$

$$\operatorname{Ind}_H^G M := \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M), \quad (g \cdot f)(x) := f(xg).$$

Lemma 2.2. *Let $S \subset G$ be a set of representatives for the right H-cosets in G. For every H-module M the map*

$$\operatorname{Ind}_H^G M \to \operatorname{ind}_H^G M, \quad f \mapsto \sum_{g \in S} g^{-1} \otimes f(g)$$

is a G-equivariant isomorphism, which is independent of S.

Proof. The independence of S follows from

$$(hg)^{-1} \otimes f(hg) = g^{-1}h^{-1} \otimes hf(g) = g^{-1} \otimes f(g)$$

for all $h \in H$ and $g \in G$ and $f \in \operatorname{Ind}_H^G M$. The G-equivariance follows from

$$\sum_{g \in S} g^{-1} \otimes (x \cdot f)(g) = \sum_{g \in S} g^{-1} \otimes f(gx) = \sum_{g' \in Sx} xg'^{-1} \otimes f(g') = x \cdot \sum_{g' \in Sx} g'^{-1} \otimes f(g')$$

for all $f \in \operatorname{Ind}_H^G M$ and $x \in G$ with the substitution $g' := gx$ and using the fact that Sx is again a set of representatives for the right H-cosets in G.

To prove that it is an isomorphism, note that by bilinearity of the tensor product, every element in $\operatorname{ind}_H^G M$ can be written as $\sum_{g \in S} g^{-1} \otimes b_g$ for certain elements $b_g \in M$. Then the map defined by

$$\sum_{g \in S} g^{-1} \otimes b_g \mapsto \left(G \ni x \mapsto \begin{cases} xg^{-1}b_g & \text{if } Hg = Hx \\ 0 & \text{otherwise} \end{cases} \right)$$

for $x \in G$ is an inverse. □

Lemma 2.3. *The functor Ind_H^G is exact and preserves injectives.*

Proof. The functor Ind_H^G is right adjoint to the restriction functor res_H^G defined by the inclusion $H \to G$ and so left exact. On the other hand ind_H^G is left adjoint to the restriction functor res_H^G and so right exact. As the two functors are isomorphic, we conclude exactness.

Let I be an injective H-module. Then the functor $\mathrm{Hom}_{\mathbb{Z}[H]}(-, I)$ is exact. Furthermore, the restriction functor res^G_H is exact. Hence the composition $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathrm{res}^G_H -, I)$ is exact, and it is isomorphic to the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(-, \mathrm{Ind}^G_H I)$ by adjointness. We conclude that $\mathrm{Ind}^G_H I$ is an injective G-module. □

Proposition 2.4 (Shapiro's Lemma)**.** *For every $q \geqslant 0$ and every H-module M there is an isomorphism $\mathrm{H}^q(G, \mathrm{Ind}^G_H M) \cong \mathrm{H}^q(H, M)$.*

Proof. By Lemma 2.3 the functor Ind^G_H is exact and preserves injectives. By the adjunction $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \mathrm{Ind}^G_H M) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, M)$ the set of G-invariants of $\mathrm{Ind}^G_H M$ is isomorphic to the set of H-invariants of M. We conclude the statement. □

Lemma 2.5. *Let $S \subset G$ be a set of representatives for the right H-cosets in G. For every G-module M the map*

$$\mathrm{Ind}^G_H M \to M, \quad f \mapsto \sum_{g \in S} g^{-1} \cdot f(g)$$

is a G-equivariant homomorphism.

Proposition 2.6. *Set $n := [G : H]$. For every $q \geqslant 0$ and every G-module M the multiplication by n map $\mathrm{H}^q(G, M) \to \mathrm{H}^q(G, M)$ factors through $H^q(H, M)$.*

Proof. The composition of $M \to \mathrm{Ind}^G_H M, \quad m \mapsto (g \mapsto gm)$ with the homomorphism of Lemma 2.5 is the homomorphism $M \to M$ given by multiplication by n. Thus the multiplication by n map $\mathrm{H}^q(G, M) \to \mathrm{H}^q(G, M)$ factors through $\mathrm{H}^q(G, \mathrm{Ind}^G_H M)$, which is isomorphic to $H^q(H, M)$ by Proposition 2.4. □

Corollary 2.7. *Let $n := |G|$. Then for all $q > 0$ and every G-module M the cohomology group $\mathrm{H}^q(G, M)$ is n-torsion.*

Proof. By Proposition 2.6 the multiplication by n map on $\mathrm{H}^q(G, M)$ factors through the group $\mathrm{H}^q(\{1\}, M) = 0$. □

2.2 Profinite Groups

Let G be a profinite group.

Definition 2.8. Let A be an abelian group. We define $\mathrm{Ind}^G A := \mathrm{colim}_U \mathrm{Ind}^{G/U}_{\{1\}} A$, where the colimit runs over all open normal subgroups $U \subset G$. Note that $\mathrm{Ind}^G A$ is equipped with a G-action.

Lemma 2.9. *Let A be an abelian group. Then for every $q \geqslant 1$ we have $\mathrm{H}^q(G, \operatorname{Ind}^G A) = 0$.*

Proof. Note that for every open normal subgroup $U' \subset G$ the set of U'-invariants satisfies

$$(\operatorname*{colim}_U \operatorname{Ind}_{\{1\}}^{G/U} A)^{U'} = \operatorname{Ind}_{\{1\}}^{G/U'} A.$$

By Proposition 3.7 of Lukas' notes it follows

$$\mathrm{H}^q(G, \operatorname{Ind}^G A) \cong \operatorname*{colim}_U \mathrm{H}^q(G/U, (\operatorname{Ind}^G A)^U) = \operatorname*{colim}_U \mathrm{H}^q(G/U, \operatorname{Ind}_{\{1\}}^{G/U} A) = 0,$$

where the vanishing follows from Proposition 2.4. □

Recall: For a prime p a group of order a power of p is called a p-group. A limit of finite p-groups is called a pro-p-group. A subgroup G_p of a profinite group G is called Sylow p-subgroup if it is closed and for every open normal subgroup $U \subset G$ the image of G_p in G/U is a Sylow p-subgroup.

Lemma 2.10. *Let p be a prime and let G be a finite p-group. Every finite p-power-torsion G-module M with $M \neq 0$ satisfies $M^G \neq 0$.*

Proof. The set $M \smallsetminus M^G$ is the disjoint union of all orbits that are of length $\geqslant 2$. The length of every such orbit must be divisible by p since G is a p-group. Hence $|M \smallsetminus M^G|$ and $|M|$ are both divisible by p. We conclude that $|M^G| > 1$. □

Lemma 2.11. *Let G be a pro-p-group. Every finite p-power-torsion G-module M admits a filtration with subquotients isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with trivial action of G.*

Proof. Because M is finite, the stabilizers are open normal subgroups of G and there are only finitely many of them. Hence there is an open normal subgroup $U \subset G$ which induces an action of the finite p-group G/U on M. By Lemma 2.10 we conclude that $M^{G/U} \neq 0$.

We use induction on $m := |M|$, which is a p-power by the assumption on M. For $m = p$ the statement is true because $M^{G/U} \neq 1$ and so $M^{G/U} = M$, hence $M \cong \mathbb{Z}/p\mathbb{Z}$ and the action is trivial. Now assume that $m > p$ and that the statement is true for every finite p-power-torsion G-module. The module $M/M^{G/U}$ is again a p-power-torsion G-module and it has cardinality stricly less than m. By the induction hypothesis, there is a filtration of $M/M^{G/U}$ with subquotients isomorphic to $\mathbb{Z}/p\mathbb{Z}$. This filtration lifts to a filtration of M which contains $M^{G/U}$ with subquotients isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Since G/U acts trivially on $M^{G/U}$ we can extend this filtration to the left by a composition series of $M^{G/U}$. □

We can now prove our main vanishing theorem of the cohomology of profinite groups:

Theorem 2.12. *Let G be a profinite group. Assume that for every prime p there is a Sylow p-subgroup $G_p \subset G$ such that $\mathrm{H}^2(G_p, \mathbb{Z}/p\mathbb{Z}) = 0$. Then $\mathrm{H}^q(G, M) = 0$ for every $q \geqslant 2$ and every torsion G-module M.*

Proof. We procede in three steps:

(a) Let p be a prime. We prove that every finite p-power-torsion G-module M satisfies $\mathrm{H}^2(G, M) = 0$. Let $G_p \subset G$ be a Sylow p-subgroup and let $U \subset G$ be an open normal subgroup. Then the index $a := [G/U : G_p/(G_p \cap U)]$ is not divisible by p. By Proposition 2.6 the multiplication by a map on $\mathrm{H}^2(G/U, M^U)$ factors through $\mathrm{H}^2(G_p/(G_p \cap U), M^U)$. But because M^U is a p-power-torsion G-module the multiplication by a map is an isomorphism. Hence the induced restriction map $\mathrm{H}^2(G/U, M^U) \to \mathrm{H}^2(G_p/(G_p \cap U), M^U)$ is injective. By taking the colimit we obtain an injective restriction map $\mathrm{H}^2(G, M) \to \mathrm{H}^2(G_p, M)$. By Lemma 2.11 there is a filtration $0 \subset M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ whose subquotients are isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Let $0 \leqslant i < \ell$. By assumption $\mathrm{H}^2(G_p, \mathbb{Z}/p\mathbb{Z}) = 0$, so by using the long exact sequence in cohomology associated to $0 \to M_i \to M_{i+1} \to \mathbb{Z}/p\mathbb{Z} \to 0$ we conclude that there is a surjection $\mathrm{H}^2(G_p, M_i) \to \mathrm{H}^2(G_p, M_{i+1})$. Since this is true for every $0 \leqslant i < \ell$ and $M_0 \cong \mathbb{Z}/p\mathbb{Z}$ we conclude that there is a surjection $0 = \mathrm{H}^2(G_p, M_0) \to \mathrm{H}^2(G_p, M)$ and hence the latter vanishes. Using the injective map $\mathrm{H}^2(G, M) \to \mathrm{H}^2(G_p, M)$ constructed above we conclude that $\mathrm{H}^2(G, M) = 0$.

(b) We prove that every torsion G-module M satisfies $\mathrm{H}^2(G, M) = 0$. We have

$$\begin{aligned} \mathrm{H}^2(G, M) &\cong \operatorname*{colim}_U \mathrm{H}^2(G/U, M^U) \\ &\cong \operatorname*{colim}_U \mathrm{H}^2(G/U, \bigoplus_p \operatorname*{colim}_r M^U[p^r]) \\ &\cong \bigoplus_p \operatorname*{colim}_r \operatorname*{colim}_U \mathrm{H}^2(G/U, M^U[p^r]) \\ &\cong \bigoplus_p \operatorname*{colim}_r \mathrm{H}^2(G, M[p^r]) \\ &= 0, \end{aligned}$$

where the vanishing follows from part (a). Here we also used that for the finite groups G/U group cohomology commutes with filtered colimits and direct sums, as can be seen by using the $\mathbb{Z}$-bar resolution to compute group cohomology.

(c) The natural injective map $M^U \to \mathrm{Ind}_{\{1\}}^{G/U} M$ passes by taking colimits to an injective homomorphism of G-modules $M \to \mathrm{Ind}^G M$. Consider the short exact sequence $0 \to M \to \mathrm{Ind}^G M \to (\mathrm{Ind}^G M)/M \to 0$. Passing to the long exact sequence and using Lemma 2.9 we conclude that $\mathrm{H}^{q-1}(G, (\mathrm{Ind}^G M)/M) \cong \mathrm{H}^q(G, M)$ for all $q \geqslant 2$. Because M is torsion, so is $\mathrm{Ind}^G M$ and so is $(\mathrm{Ind}^G M)/M$. By using (b) and an induction on q we conclude that $\mathrm{H}^q(G, M) = 0$ for all $q \geqslant 2$. □

We can get rid of the assumption that M is torsion by paying with one cohomological degree:

Corollary 2.13. *Let G be a profinite group. If for every prime p there exists a Sylow p-subgroup G_p of G such that $\mathrm{H}^2(G_p, \mathbb{Z}/p\mathbb{Z}) = 0$, then $\mathrm{H}^q(G, M) = 0$ for all $q \geqslant 3$ and every G-module M.*

Proof. Note that for every open normal subgroup $U \subset G$ the group $\mathrm{H}^q(G/U, M^U \otimes \mathbb{Q})$ is torsion for $q \geqslant 1$ by Corollary 2.7 but also free because we tensored by $\mathbb{Q}$. Hence these cohomology groups are zero. We conclude that $\mathrm{H}^q(G, M \otimes \mathbb{Q}) \cong \mathrm{colim}_U \mathrm{H}^q(G/U, M^U \otimes \mathbb{Q})$ is zero, too. Consider the exact sequence

$$0 \to M_{\mathrm{tors}} \to M \to M \otimes \mathbb{Q} \to M \otimes \mathbb{Q}/\mathbb{Z} \to 0,$$

which we split into two:

$$0 \to M_{\mathrm{tors}} \to M \to N \to 0, \qquad 0 \to N \to M \otimes \mathbb{Q} \to M \otimes \mathbb{Q}/\mathbb{Z} \to 0.$$

Note that $M \otimes \mathbb{Q}/\mathbb{Z}$ is torsion, so by Theorem 2.12 its cohomology groups in degree $q \geqslant 2$ vanish. By a long exact sequence we obtain isomorphisms $\mathrm{H}^q(G, N) \cong \mathrm{H}^q(G, M \otimes \mathbb{Q}) = 0$ for $q \geqslant 3$. By another long exact sequence and the same argument we therefore obtain isomorphisms $\mathrm{H}^q(G, M) \cong \mathrm{H}^q(G, N) = 0$ for all $q \geqslant 3$. □

3 Vanishing in Étale Cohomology

We state the following proposition without proof:

Proposition 3.1 (Artin-Schreier sequence)**.** *Let p be a prime. For every scheme X over $\mathbb{F}_p$ the natural sequence of étale sheaves*

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{x \mapsto x - x^p} \mathbb{G}_a \to 0$$

is exact.

Lemma 3.2. *For every field K and every $q \geqslant 1$ we have $\mathrm{H}^q(\mathrm{Spec}(K), \mathbb{G}_a) = 0$.*

Proof. Let K^s be a separable closure of K and let G be the Galois group of K^s over K. Let $U \subset G$ be an open normal subgroup and let L be the corresponding finite Galois extension of K. By the normal basis theorem there exists an element $x \in L$ such that $(\sigma(x))_{\sigma \in G}$ is a K-basis of L. We obtain a group homomorphism

$$L \ni \sum_{\sigma \in G} k_\sigma \sigma(x) \mapsto (\tau \mapsto k_{\tau^{-1}}) \in \mathrm{Ind}_{\{1\}}^{G/U} K.$$

This is in fact an isomorphism of G/U-modules. We conclude that $\mathrm{colim}_L \mathbb{G}_a(L) = \mathrm{colim}_L L \cong \mathrm{Ind}^G K$, where the colimit runs over all finite Galois extensions of K in K^s. Using Proposition

2.7 of Lukas' talk and Lemma 2.9 we obtain for all $q \geqslant 1$:

$$\mathrm{H}^q(\mathrm{Spec}(K), \mathbb{G}_a) \cong \mathrm{H}^q(G, \operatorname*{colim}_L \mathbb{G}_a(L)) \cong \mathrm{H}^q(G, \mathrm{Ind}^G K) = 0.$$

□

Now we can state our main vanishing theorem:

Theorem 3.3. *Let K be a field. If $\mathrm{H}^2(\mathrm{Spec}(L), \mathbb{G}_\mathrm{m}) = 0$ for every separable algebraic extension L/K then $\mathrm{H}^q(\mathrm{Spec}(K), \mathbb{G}_\mathrm{m}) = 0$ for every $q \geqslant 1$.*

Proof. The case $q = 1$ is Hilbert 90 (see Prop. 4.1. in Lukas' notes). The cohomology group $\mathrm{H}^2(\mathrm{Spec}(K), \mathbb{G}_\mathrm{m})$ is zero by assumption. Let $q \geqslant 3$. We verify the conditions of Corollary 2.13. Let K^s be a separable closure of K and denote by G the Galois group. For every prime p pick a Sylow p-subgroup G_p of G and let K_p be the corresponding field extension of K inside K^s. We will prove that $\mathrm{H}^2(G_p, \mathbb{Z}/p\mathbb{Z}) = 0$ for every prime p. There are two cases:

(a) Assume that $\mathrm{char}(K) \neq p$. Then we can use the long exact sequence induced by the Kummer sequence

$$1 \to \mu_p \to \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m} \to 1$$

on $\mathrm{Spec}(K_p)$ to deduce that $\mathrm{H}^2(\mathrm{Spec}(K_p), \mu_p) = 0$. Indeed $\mathrm{H}^1(\mathrm{Spec}(K_p), \mathbb{G}_\mathrm{m})$ vanishes by Hilbert 90 and $\mathrm{H}^2(\mathrm{Spec}(K_p), \mathbb{G}_\mathrm{m})$ vanishes by assumption. On $\mathrm{Spec}(K_p)$ we have an isomorphism of étale sheaves $\mathbb{Z}/p\mathbb{Z} \cong \mu_p$; this can be checked on stalks. We conclude that $\mathrm{H}^2(G_p, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{H}^2(\mathrm{Spec}(K_p), \mathbb{Z}/p\mathbb{Z}) = 0$.

(b) Assume that $\mathrm{char}(K) = p$. Then we use the long exact sequence coming from the Artin Schreier sequence of Proposition 3.1

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \to \mathbb{G}_a \to 0$$

to deduce that $\mathrm{H}^2(\mathrm{Spec}(K_p), \mathbb{Z}/p\mathbb{Z}) = 0$. Indeed $\mathrm{H}^q(\mathrm{Spec}(K_p), \mathbb{G}_a) = 0$ for $q \geqslant 1$ by Lemma 3.2, so $\mathrm{H}^2(G_p, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{H}^2(\mathrm{Spec}(K_p), \mathbb{Z}/p\mathbb{Z}) = 0$.

We see that the assumptions of Corollary 2.13 are satisfied and conclude together with the case of $q = 1$ and $q = 2$ above that $\mathrm{H}^q(\mathrm{Spec}(K), \mathbb{G}_\mathrm{m}) = 0$ for all $q \geqslant 1$. □

By Corollary 1.5 we obtain

Corollary 3.4. *Let K be a field of transcendence degree 1 over an algebraically closed field. Then $\mathrm{H}^q(\mathrm{Spec}(K), \mathbb{G}_\mathrm{m}) = 0$ for every $q \geqslant 1$.*

References

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