Higher Direct Image of Constructible Sheaves

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The goal of this talk is to prove the following Theorem:

Theorem 0.1 ([1] XIV Thm 1.1) *Let $f : X \to S$ be a proper morphism and $\mathcal{F}$ a constructible torsion sheaf on X. Then for all $q \geqslant 0$, the higher direct image $R^q f_* \mathcal{F}$ is a constructible sheaf.*

1 Reduction to the projective case

Proposition 1.1 *Let $f : X \to S$ be a finite morphism. Then the higher direct image functors $R^q f_*$ preserve constructible sheaves.*

To prove this, we need the following criterion for a sheaf to be constructible.

Proposition 1.2 ([1] IX 2.13 (iii)) *Let X be a noetherian scheme and $\mathcal{F}$ an abelian étale sheaf with finite stalks. Let*

$$c : X \to \mathbb{Z}, \quad x \mapsto |\mathcal{F}_{\bar{x}}|$$

where $\bar{x}$ is a geometric point of X lying above x. Then $\mathcal{F}$ is constructible if and only if for all $n \in \mathbb{Z}$, the preimage $c^{-1}(n)$ is constructible.

Proof. It suffices to show that the proposition is true in the case that c is constant. Let $V \to X$ be an étale neighborhood of a geometric point of X such that $\mathcal{F}(V)$ which surjects onto $\mathcal{F}_P$. Let $S \subset \mathcal{F}_P$ be a subst such that S maps bijectively $\mathcal{F}_P$. For any specialization morphism $P \to Q$ of geometric points, we have a commutative diagram

$$\begin{tikzcd}
(S_V)_P \arrow[r, "\cong"] & \mathcal{F}_P \\
(S_V)_Q \arrow[u, "\cong"] \arrow[r] & \mathcal{F}_Q. \arrow[u]
\end{tikzcd}$$

Since $c(\mathcal{F}_Q) = c(\mathcal{F}_Q) < \infty$, the map $\mathcal{F}_Q \to \mathcal{F}_P$ is a bijection. Hence $\mathcal{F}$ is locally constant at P. By noetherian induction we are done.

□

Proof. [Proof of Prop 1.1] We will apply Proposition 1.2. For this we will compute the stalks of $R^q f_* G_X$. Since for $q \geqslant 1$, the higher direct images vanish, it suffices to compute the stalks of $f_*\mathcal{F}$. We choose a strata $\{\pi_i : X_i \to X\}$ and constant sheaves C_i on X_i. Then we have an injection $\mathcal{F} \to \prod_i \pi_i C_i$. Since the direct image functor is left exact, this gives an injective morphism

$$f_*\mathcal{F} \to \prod_i (f \circ \pi_i)_* C_i.$$

By [1] IX 2.6, it thus suffices to show that $\prod_i (f \circ \pi_i)_* C_i$ is constructible. The stalk at $s \in S$ is

$$(f_* G_X)_s = G^{\text{ number of connected components of } X_s}.$$

The following Lemma concludes the proof. □

Lemma 1.3 ([2] 9.7.9.) *Let $f : X \to S$ be a morphism of finite presentation. Then the function*

$$S \to \mathbb{Z}, \quad s \mapsto \textit{number of connected components of } X_s$$

is constructible.

Proposition 1.4 (Chow's Lemma) *Let S be a noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a commutative diagram*

$$\begin{tikzcd} X \arrow[dr] & \bar{X} \arrow[l, "\pi"'] \arrow[r] \arrow[d, "f"] & \mathbb{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

consisting of a proper and surjective morphism $\pi : \bar{X} \to X$ and a closed embedding $\bar{X} \to \mathbb{P}^n_S$ such that π is generically an isomorphism on the target, i.e. there exists an open dense subset $U \subset X$ such that $U \times_X \bar{X} \to U$ is an isomorphism.

Proposition 1.5 (Domination Trick) *Let S be noetherian. Then f satisfies Theorem 0.1 if π, $\bar{f}$ and $f|_{X \setminus U}$ satisfy the Theorem.*

Proof. Let $j : U \to X$ denote an open subset such that $U \times_X \bar{X} \to U$ is an isomorphism. Let $\iota : X \setminus U \to X$ be a the canonical inclusion. Then we have a short exact sequence

$$0 \to j_* j^* \mathcal{F} \to \mathcal{F} \to \iota_* \iota^* \mathcal{F} \to 0$$

as can be seen on the stalks. Since the pull-back of a constructible sheaf is constructible, the sheaf $\iota^*\mathcal{F}$ is constructible and by Prop 1.1, the sheaf $\iota_*\iota^*\mathcal{F}$ is constructible. Since the subcategory of constructible sheaves is Serre, the sheaf $j_* j * \mathcal{F}$ is constructible.

Using the long exact sequence associated to this short exact sequence and the fact that constructible sheaves are Serre, we see that to show that $R^p f_* \mathcal{F}$ is constructible it suffices to show that both $R^p f_* j_* j^* \mathcal{F}$ and $R^p f_* \iota_* \iota^* \mathcal{F}$ are constructible. Since ι is acyclic, we have

$$R^p f_*(\iota_* \iota^* \mathcal{F}) \cong R^p (f \circ \iota)_* \iota^* \mathcal{F}.$$

By assumption on $f \circ \iota$, the right hand side is constructible, thus so is the left hand side. We denote $\bar{j} : \bar{U} := U \times_X \bar{X} \to \bar{X}$ and $\bar{\mathcal{F}} = \pi^* \mathcal{F}$. Then by the Lerray spectral sequence we have

$$(R^p f_*)(R^q \pi_*)(\bar{j}_* \bar{j}^* \bar{\mathcal{F}}) \Rightarrow R^{p+q} \bar{f}_* (\bar{j}_* \bar{j}^* \bar{\mathcal{F}}).$$

Since π is an isomorphism above $X \backslash Y$, we have

$$R^p \pi_* \mathcal{F} = 0 \text{ for } q > 0.$$

Hence we get

$$R^p f_*(\pi_* \bar{j}_* \bar{j}^* \bar{\mathcal{F}}) \cong R^p \bar{f}_* (\bar{j}_* \bar{j}^* \bar{\mathcal{F}}).$$

Furthermore, we have $\pi_* \bar{j}_* \bar{j}^* \bar{\mathcal{F}} \cong j_* j^* \mathcal{F}$ and thus $R^p f_*(j_* j^* \mathcal{F}) \cong R^p \bar{f}_* (\bar{j}_* \bar{j}^* \bar{\mathcal{F}})$. By assumption on $\bar{f}$, the right hand side is constructible, and thus so is the left hand side. Thus $R^p f_*$ preserves constructible sheaves. □

Proposition 1.6 *To show that $f|_{X \backslash U}$ satisfies the Theorem, it suffices to show that all projective morphisms satisfy the Theorem.*

Proof. Since $f \circ \pi = \bar{f}$, and $\bar{f}$ is projective and f proper, we know that π is projective. Thus it remains to show that the Theorem is true for $f|_{X \backslash U}$ if the Theorem is true for all projective morphisms.

Denote $X \backslash U =: X_1$ and $f_1 := f|_{X \backslash U}$. The morphism f_1 is proper. By Chows Lemma, there exists an S-scheme $\bar{X}_1$, a projective morphism $\bar{f}_1$ and a projective morphism $\pi : \bar{X}_1 \to X_1$ such that the diagram

$$\begin{tikzcd} X_1 \arrow[dr] & \bar{X}_1 \arrow[l] \arrow[d] \\ & S \end{tikzcd}$$

commutes. In fact there exits a $U_1 \subset X_1$ such that $U_1 \times_{X_1} \bar{X}_1 \to U_1$ is an isomorphism. Thus, by the Domination trick, and if all projective morphism satisfy the Theorem, it suffices to show that the for $X_2 := X_1 \backslash U_1$ the morphism $f_2 := f_1|_{X_2} : X_2 \to S$ satisfies the Theorem. By Noetherian induction, we may therefore assume that for some $n \geqslant 0$, the morphism f_n is projective. Thus we have now reduced ourself to the case that $f : X \to S$ is projective. □

Proposition 1.7 *Let $f : U \to V$ and $g : V \to W$ be morphisms such that the higher direct image functors $R^q f_*$ and $R^q g_*$ preserve constructible sheaves. Then the higher direct image functor $R^q(g \circ f)_*$ also preserves constructible sheaves.*

Proof. Let $\mathcal{F}$ be a constructible sheaf on U. Then we have the Lerray spectral sequence

$$R^p g_* R^q f_* \mathcal{F} \Rightarrow R^{p+q}(g \circ f)_* \mathcal{F}.$$

This yields a filtration of $R^n(g \circ f)_* \mathcal{F}$ where the quotients are constructible sheaves. Since the subcategory of constructible sheaves are Serre, the sheaf $R^n(g \circ f)_* \mathcal{F}$ is constructible. $\square$

Proposition 1.8 *To show Theorem 0.1, it suffices to show the structure morphism $\mathbb{P}^1_S \to S$ satisfy Theorem 0.1.*

Proof. Let $X \to S$ be a projective morphism of fiberwise dimension n. Assume that the Theorem is true for $\mathbb{P}^1_S \to S$. We will show by induction on the fiberwise dimension of $X \to S$ that the Theorem is true. After possibly shrinking S, there exits a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\ S & \end{tikzcd}$$

such that $X \to \mathbb{P}^n_S$ is finite. By Prop 1.7 the composition of two morphisms who satisfy the Theorem satisfies the Theorem, and since finite morphisms satisfy the Theorem, we have reduced ourself to the case that f is the structure morphism $\mathbb{P}^n_S \to S$. Thus the case that $n = 1$ is done.

Assume that every projective morphism of fiberwise dimension $< n$ satisfies the Theorem. Let $\bar{X}$ be the blow-up of $\mathbb{P}^n_S$ at $\mathbb{P}^{n-2}_S$. Then we have a commutative diagram

$$\begin{tikzcd} & \bar{X} \arrow[dl] \arrow[dr, "a"] \arrow[dd, "\bar{f}"] & \\ \mathbb{P}^n_S \arrow[dr, "f"'] & & \mathbb{P}^1_S \arrow[dl] \\ & S & \end{tikzcd}.$$

Note that the morphism a is of fiberwise dimension $\leqslant n-1$. Hence by induction assumption, the morphisms $\bar{X} \xrightarrow{a} \mathbb{P}^1_S$ and $\mathbb{P}^1_S \to S$ satisfy the Theorem. Hence the morphism $\bar{X} \xrightarrow{\bar{f}} S$ satisfies the Theorem. Thus by the Domination trick, the morphism $\mathbb{P}^n_S \to S$ satisfies the Theorem. $\square$

2 Proof of the projective case

It remains to show the following special case.

Proposition 2.1 *Let $f : \mathbb{P}^1_S \to S$ be the structure morphism. The higher direct image functors $R^q f_*$ preserve constructible sheaves.*

Proposition 2.2 *Proposition 2.1 is true if the Theorem is true for all projective S-schemes of relative dimension one $f : X \to S$ and constant torsion sheaves $\mathbb{Z}/(l)$.*

Proof. Let $\mathcal{F}$ be a constructible sheaf on $\mathbb{P}^1_S$. Then there exists a stata

$$\{\pi_i : X_i \to \mathbb{P}^1_S \text{ finite} \mid i \in I\}$$

and for each $i \in I$ an n_i such that we have an injective morphism

$$\mathcal{F} \to \bigoplus_{i \in I} \pi_{i*}\mathbb{Z}/(n_i).$$

Thus there exits a resolution $\mathcal{F} \to \mathcal{G}^\bullet$ where $\mathcal{G}^p = \bigoplus \pi_{i,p*}\mathbb{Z}/(n_{i,p})$. Since finite morphism are acyclic, we get a spectral sequence

$$R^q(\pi \circ \pi_{i,p})_*\mathbb{Z}/(n_{i,p}) \Rightarrow R^{p+q}\pi_*\mathcal{F}.$$

It suffices to show that all $\pi \circ \pi_{i,p}$ satisfy the Theorem, since then we have a filtration of $R^{p+q}\pi_*\mathcal{F}$ who's quotient consist of constructible sheaves. But since the subcategory of constructible sheaves is Serre, this means that $R^{p+q}\pi_*\mathcal{F}$ is also constructible. □

Proposition 2.3 *Proposition 2.2 is true if the Theorem is true for all smooth and projective curves $X \to S$ and constant torsion sheaves.*

Proof. We may assume that S is of finite type over $\mathbb{Z}$, affine and integral. Denote $S = \operatorname{Spec} R$ and let ν be the generic point. By [2] 17.15.14. , there exists a finite field extension $K'/k(\nu)$ such that there exits a projective smooth $k(\nu)$-scheme $\tilde{X}_\nu$, a finite surjection $\tilde{X}_\nu \to X_\nu := X \times_R k(\nu)$. Denote by X_i the irreducible components of X_ν^{red}. Then we have a construction

$$X_\nu \leftarrow X_\nu^{\mathrm{red}} \leftarrow \sqcup X_i \leftarrow \sqcup \tilde{X}_i =: \tilde{X}_\nu.$$

Now let R' be the integral closure of R in K'. Since R is of finite type over $\mathbb{Z}$, the ring R' is finite over R. Then there exists an $r \in R$ and $\mathfrak{X} \to \operatorname{Spec} R'[\frac{1}{r}]$ a projective smooth such that $\mathfrak{X}_\nu = \tilde{X}_\nu$. Then since $R[\frac{1}{r}] \to R'[\frac{1}{r}]$ is finite, it suffices to show the Theorem for $R'[\frac{1}{r}]$. We want to show that there exists an R-morphism $\mathfrak{X} \to X$. We have a cartesian diagram

$$\begin{tikzcd} X_K^{\mathrm{norm}} \arrow[r] \arrow[d] & X_K \arrow[r] & \mathbb{P}^n_K \arrow[d] \\ \mathfrak{X} \arrow[rr] & & \mathbb{P}^n_R \end{tikzcd}$$

We will show that the morphism $\mathfrak{X} \to \mathbb{P}^n_R$ factors through $X \to \mathbb{P}^n_R$. By construction, on the generic fiber of R, we know that the image of $\mathfrak{X}_\nu$ is contained in the image of X_ν. The ideal sheaves of the images are $I, J \subset \mathcal{O}_{\mathbb{P}^n_R}$ are thus such that $I|_\nu \subset J|_\nu$. These ideal sheaves are finitely generated, thus there exist finitely many elements of $k(\nu)$ which are needed to include any element of $I|_\nu$ into $J|_\nu$. Thus, by localizing at those elements, we see that there exists an open neighborhood U of ν such that $I|_U \subset J|_U$. Thus, after shrinking $S = \operatorname{Spec} R$ and renaming it R, we may assume that $\mathfrak{X}$ factors through X. Then we have a commutative diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[dr] & X \arrow[d] \\ & \operatorname{Spec} R. \end{tikzcd}$$

where $\mathfrak{X} \to \operatorname{Spec} R$ is smooth and projective, $X \to \operatorname{Spec} R$ is projective and thus the morphism $\mathfrak{X} \to X$ is also projective, surjective and thus quasi-finite.

There exists an open locus in $\mathfrak{X}$ such that the morphism $\mathfrak{X} \to X$ is finite. Since $\mathfrak{X} \to X$ is projective, we can shrink $\operatorname{Spec} R$ again and consider the fiber-products of $\mathfrak{X}$ and X above this open subset of $\operatorname{Spec} R$, such that the morphism $\mathfrak{X} \to X$ is finite. Then we can apply the Domination Trick Prop 1.5, and thus it suffices to show that the morphism $\mathfrak{X} \to \operatorname{Spec} R$ satisfies the Theorem. By construction, this morphism is smooth, projective and fiberwise of dimension one and thus it suffices to show the Theorem for morphisms which are smooth, projective and fiberwise of dimension one. □

Proposition 2.4 *The Theorem is true for all smooth and projective curves $X \to S$ and constant torsion sheaves.*

Proof. By the Proper Base Change we may assume that k is separably closed. To show that $R^p f_* \mathcal{F}$ is constructible, we want to use Prop 1.2, i.e. we have to show that the function

$$S \to \mathbb{Z}, \quad s \mapsto |(R^p f_* \mathcal{F})_s|$$

is constructible. By Proper Base Change on the diagram

$$\begin{tikzcd} X_s \arrow[r] \arrow[d, "f_s"] & X \arrow[d] \\ \operatorname{Spec} k(s) \arrow[r] & S \end{tikzcd}$$

we have $(R^p f_* \mathbb{Z}/(l))_s = R^p f_{s*} \mathbb{Z}/(l)_s$. By a further application of the Proper Base Change Theorem, we know that $R^p f_* \mathbb{Z}/(l))_s = H^q(X_s, \mathbb{Z}/(l))$ where we may assume that $k(s)$ is separably closed. Let us compute these cohomology groups. There are two cases, namely either l does or does not divide the characteristic of k. If $l \nmid \mathrm{char}(k)$, then

$$H^q(X_s, \mathbb{Z}/(l)) = \left\{ \begin{array}{ll} (\mathbb{Z}/(l))^{\text{number of connected components of } X_s}, & \text{for } q = 0 \\ (\mathbb{Z}/(l))^{2g \cdot \text{number of connected components of } X_s}, & \text{for } q = 1 \\ (\mathbb{Z}/(l))^{\text{number of connected components of } X_s}, & \text{for } q = 2 \\ 0, & \text{for } q > 2 \end{array} \right\}$$

where g is the genus of X_s. Since the morphism $X \to S$ is smooth, the genus of the fiber X_s is locally constant. By Lemma 1.3, the number of connected components is constructible. Thus the function $s \mapsto |H^q(X_s, \mathbb{Z}/(l))|$ is constructible. The computation in the case of the bad primes uses Artin-Schreier, and we do not do it right now. However, the statement is true. $\square$

References

[1] M. Artin, A. Grothendieck, J.-L.Verdier, SGA IV Thorie des topos et cohomologie étale des schémas. Tomes 1à 3

[2] A. Grothendieck, Éléments de géométrie algébique IV, Étude locale des schémas et des morphismes de schêmas, Troisième partie