Local Henselian Rings—For the Étale Cohomology Seminar.
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Contents

The reference for everything that follows is the Stacks Project [stacks], to which all the "Tag" hyperlinks lead. All our diagrams are commutative unless otherwise specified. We require separable field extensions to be algebraic.

1 Preliminaries

We recall a few definitions. A *geometric point* $\bar{y}$ of a scheme Y is a morphism $\bar{y}\colon \operatorname{Spec} k \to Y$, such that k is separably closed. We shall often denote by y the unique point $y \in Y$ in the image of $\bar{y}$.

An *étale neighbourhood* of a geometric point $\bar{y}$ on Y is a pair $(V, \bar{v})$ of an étale morphism $V \to Y$ and a geometric point $\bar{v}$ with $\kappa(\bar{v}) = k$ of V such that the composition $\operatorname{Spec} k \to V \to Y$ is $\bar{y}$. We will denote the category of all étale neighbourhoods of $\bar{y}$ by $\mathscr{N}_{\bar{y}}$. We will denote the subcategory of all *affine* étale neighbourhoods of $\bar{y}$ by $\mathscr{N}_{\bar{y}}^{\mathrm{aff}}$.

Let $\mathscr{F}$ be a sheaf on $Y_{\text{ét}}$. Recall that the *stalk* of $\mathscr{F}$ at a geometric point $\bar{y}$ on Y is the following colimit over étale neighbourhoods of $\bar{y}$

$$\begin{aligned} \mathscr{F}_{\bar{y}} &:= \operatorname*{colim}_{(V,\bar{v}) \in \mathscr{N}_{\bar{y}}} \mathscr{F}(V) \\ &= \operatorname*{colim}_{(V,\bar{v}) \in \mathscr{N}_{\bar{y}}^{\mathrm{aff}}} \mathscr{F}(V), \end{aligned}$$

which we may refine as a colimit over affine étale neighbourhoods of $\bar{y}$.

2 The main result

The main result of this talk is as follows.

Main Result 2.1 (Tag 03QP)**.** *If $f\colon X \to Y$ is a finite morphism of schemes, then*

$$R^q f_* \mathscr{F} = 0,$$

for all $q \geq 1$ and all $\mathscr{F} \in \operatorname{Ab} X_{\acute{e}t}$.

Attempt at a proof. It is enought to show vanising at the stalks, i.e. we want to show

$$(R^q f_* \mathscr{F})_{\bar{y}} = 0,$$

for all geometric points $\bar{y}$ on Y. Our proof will be resting on three observations.

Obs. 1. The sheaf $R^q f_* \mathscr{F} \in \mathrm{Ab}\, Y_{\text{ét}}$ is the sheafification of the presheaf

$$(V \xrightarrow{\text{ét}} Y) \mapsto H^q_{\text{ét}}(X \times_Y V, \mathscr{F}|_{X \times_Y V}).$$

(Work it out by hand or see [stacks, Tag 03Q8].) In particular, as sheafification doesn't change the stalks, we may compute the stalks as

$$(R^q f_* \mathscr{F})_{\bar{y}} = \operatorname*{colim}_{(V,\bar{v}) \in \mathscr{N}_{\bar{y}}} H^q_{\text{ét}}(X \times_Y V, \mathscr{F}|_{X \times_Y V}),$$

where as in the Preliminaries 1, $\mathscr{N}_{\bar{y}}$ denotes the category of étale neighbourhoods of $\bar{y}$.

Obs. 2. We have that

$$\lim_{(V,\bar{v}) \in \mathscr{N}_{\bar{y}}} X \times_Y V = \operatorname{Spec} \mathcal{O}_{Y,\bar{y}}$$

(See [stacks, Tag 01YW].) Now by [stacks, Tag 03Q6], we get that

$$(R^q f_* \mathscr{F})_{\bar{y}} = H^q_{\text{ét}}(X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\bar{y}}, \mathscr{F}|_{X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\bar{y}}}).$$

Obs. 3. Since $X \to Y$ is finite, so is $X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\bar{y}} \to \operatorname{Spec} \mathcal{O}_{Y,\bar{y}}$. Thus,

$$X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\bar{y}} = \operatorname{Spec} B,$$

where B is a finite $\mathcal{O}_{Y,\bar{y}}$ algebra.

Combining the three observations, we are done, conditional on the following Key Lemma 2.2 □

Key Lemma 2.2. *If B is a finite $\mathcal{O}_{Y,\bar{y}}$ algebra, and $\mathscr{G}$ is an abelian sheaf on $(\operatorname{Spec} B)_{\acute{e}t}$, then*

$$H^q_{\acute{e}t}(\operatorname{Spec} B, \mathcal{G}) = 0,$$

for all $q \geq 1$.

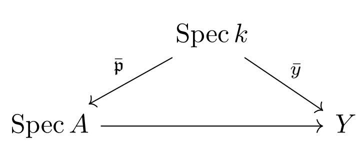
The proof of the Key Lemma is the ultimate goal of the rest of these notes. It will follow from the fact that this is true when replacing $\mathcal{O}_{Y,\bar{y}}$ by any "strictly henselian" local ring, and the fact that $\mathcal{O}_{Y,\bar{y}}$ is the "strict henselisation" of the Zariski stalk $\mathcal{O}_{Y,y}$.

3 Interlude: stalks of the structure sheaf

Before we go on to develop the theory of henselian local rings, we take a closer look at the stalks $\mathcal{O}_{Y,\bar{y}}$ of the structure sheaf. We have, by definition,

$$\mathcal{O}_{Y,\bar{y}} = \operatorname*{colim}_{(V,\bar{v}) \in \mathscr{N}_{\bar{y}}} \mathcal{O}_Y(V \to Y) = \operatorname*{colim}_{(V,\bar{v}) \in \mathscr{N}_{\bar{y}}} \mathcal{O}_V(V)$$

To better understand this, we restrict to an affine neighbourhood $\operatorname{Spec} A$ of the point y under $\bar{y}$. This doesn't change the stalk. We write $\mathfrak{p}$ for the point of $\operatorname{Spec} A$ over y, and we write $\bar{\mathfrak{p}}$ for the map $\operatorname{Spec} k \to \operatorname{Spec} A$

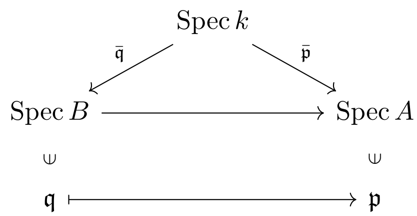


Note that the Zariski stalk $\mathcal{O}_{Y,y}$ at y is equal to $A_{\mathfrak{p}}$. We may compute $\mathcal{O}_{Y,\bar{y}}$ as

$$\mathcal{O}_{Y,\bar{y}} = \mathcal{O}_{\operatorname{Spec} A,\bar{\mathfrak{p}}} = \operatorname*{colim}_{(V,\bar{v})\in\mathcal{N}_{\bar{\mathfrak{p}}}^{\mathrm{aff}}} \mathcal{O}_V(V),$$

with the colimit taken over affine étale neighbourhoods as in the Preliminaries 1.

We have reduced to the affine situation and it is time to translate it into commutative algebra. The data of an affine étale neighbourhood $(V = \operatorname{Spec} B, \bar{v} = \bar{\mathfrak{q}})$ of $\bar{\mathfrak{p}}$ in $\operatorname{Spec} A$ is represented by a diagram



This is equivalent to giving 1) and étale algebra $A \to B$, 2) a prime $\mathfrak{q}$ over $\mathfrak{p}$ (which gives a field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ of residue fields), and 3) a $\kappa(\mathfrak{p})$ algebra morphism $\varphi\colon \kappa(\mathfrak{q}) \to k$. Finally, we compute

$$\mathcal{O}_{Y,\bar{y}} = \operatorname*{colim}_{(B,\mathfrak{q},\varphi)} B.$$

We will see that the colimit over these objects, after localising at $\mathfrak{p}$, is the strict henselisation of $A_{\mathfrak{p}}$. Moreover, when developing the theory of henselian local rings below, it might be beneficial to keep the following in mind: they will be characterised both by a) the structure of finite algebras over them (cf. Key Lemma 2.2), as well as b) the behaviour of certain étale algebras over them (cf. the stalk description above). In this way, they help us prove the Key Lemma 2.2 and with it, the Main Result 2.1.

4 Henselian local rings

4.1 Definition and alternative characterisations

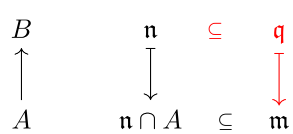
Definition 4.1. A local ring $(A, \mathfrak{m}, k)$ is *henselian* if every finite algebra $A \to B$ is a product $\prod_i B_i$ of local rings $(B_i, \mathfrak{n}_i)$. If, moreover, k is separable closed, we say that A is *strictly henselian.*

We may immediately streangthen this definition without losing generality.

Lemma 4.2. *If $(A, \mathfrak{m}, k)$ is henselian and $A \to B$ is a finite algebra, then a) B is a* finite *product of local rings $B_1, \dots, B_r$, b) the factors B_i are henselian, and c) if A is strictly henselian then so are the B_i.*

Proof. We first show that if $\mathfrak{n}$ is maximal in B, then it lies over $\mathfrak{m}$ (i.e. $\mathfrak{n} \cap A = \mathfrak{n}^c = \mathfrak{m}$).

We have that $n \cap A \subseteq \mathfrak{m}$, since otherwise $\mathfrak{n} \cap A$ would be all of A (it would contain a unit), and then $\mathfrak{n}$ would have to be all of B. The situation is thus as follows

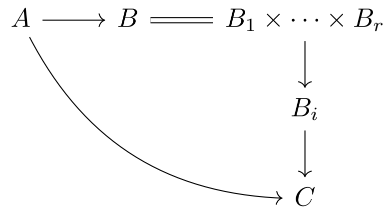


By the going up theorem (see e.g. [AM69, Prop. 5.7.]), there exists a $\mathfrak{q}$ lying over $\mathfrak{m}$, but since $\mathfrak{n}$ is maximal, $\mathfrak{n} = \mathfrak{q}$. Thus, $\mathfrak{n} \cap A = \mathfrak{m}$.

Next, note that finite implies quasi-finite (see e.g. [AM69, Exercise 4, Ch. 8]), and thus there are only finitely many maximal ideals $\mathfrak{n}$ in B.

Finally, the maximal ideals $\mathfrak{n}_i$ of the factors B_i of B give rise to distinct maximal ideals $\mathfrak{n}_i \times \prod_{j\neq i} B_j$. Putting everything together, we conclude (a).

To see that B_i is henselian, take a finite algebra $B_i \to C$, and consider the following diagram.



We conclude (b) by noting that the long arrow is finite (as it's the composition of finite morphisms), and thus C splits into a product of local rings.

For (c), we assume that k is separably closed. We have that the composite morphisms $A \to B_i$ are finite, so we get finite extensions $\kappa(\mathfrak{n}_i)/k$, and therefore $\kappa(\mathfrak{n}_i)$ are separably closed: if L is an extension with separable degree $l = [L : \kappa(\mathfrak{n}_i)]_s$ over $\kappa(\mathfrak{n}_i)$, then $[L : k]_s = 1$ and $[\kappa(\mathfrak{n}_i) : k]_s = 1$, as k is separably closed. Thus, by the multiplicativity of separable degrees (see [stacks, Tag 09HK]), $l = 1$, and hence $\kappa(\mathfrak{n}_i)$ is separably closed. □

Next, we have alternative characterisations of henselian rings. We use bars to denote reduction mod $\mathfrak{m}$.

Proposition 4.3. *If $(A, \mathfrak{m}, k)$ is a local ring then the following are equivalent:*

1. *A is henselian,*
2. *for every monic $f \in A[T]$, and every simple root $a_0 \in k$ of $\bar{f} \in k[T]$, there exists a lift $a \in A$ (so $\bar{a} = a_0$) such that $f(a) = 0$,*
3. *for every étale algebra $A \to B$, and $\mathfrak{q} \subset B$ over $\mathfrak{m}$ with $k = \kappa(\mathfrak{q})$, there exists a section $\tau \colon B \to A$ with $\tau^{-1}(\mathfrak{m}) = \mathfrak{q}$.*

Proof. See [stacks, Tag 04GG]. □

The second characterisation is the first indication of a theme we'll see more of, namely much of the behaviour of a henselian ring A being determined by behaviour at its residue field, or more geometrically, its behavour is determined at its closed point $\mathfrak{m}$. In Theorem 5.1, this will manifest étale-topologically as the spectrum of a strictly henselian ring being "contractible". The third characterisation is key in the proof of this theorem.

Recall that $(A, \mathfrak{m})$ is a *complete* local ring if the canonical morphism $A \to \lim_n A/\mathfrak{m}^n$ is an isomorphism.

Lemma 4.4 (Hensel's lemma)**.** *If $(A, \mathfrak{m}, k)$ is complete, then it is henselian.*

Reminder on Newton's method. Given some differentiable function $f \colon \mathbb{R} \to \mathbb{R}$, we have successive approximations of a root, converging rapidly, as follows. Make a guess $a_0 \in \mathbb{R}$. Then define a_{i+1}, iteratively as

$$a_{i+1} = a_i - \frac{f(a_i)}{f'(a_i)}.$$

We then get an actual root of f as the limit $\lim_{i\to\infty} a_i$. The proof of Hensel's lemma is by adapting this method.

Proof of Hensel's lemma. We show that condition 2. in Prop. 4.3 is fulfilled by A. Let $f \in A[T]$ be monic, let $f_n \in (A/\mathfrak{m}^{n+1})[T]$ be the reduction of f mod $\mathfrak{m}^{n+1}$, and let f'_n be the derivative of f. We suppose we have a simple root a_0 in $k = A/\mathfrak{m}$ of $\bar{f} = f_0$. This corresponds to the initial guess in Newton's method.

We would like to define $a_{n+1} \in A/\mathfrak{m}^{n+2}$ iteratively as in Newton's method. Thus, we suppose we're given $a_n \in A/\mathfrak{m}^{n+1}$, such that $a_0 = a_n \mod \mathfrak{m}$. Take any $b \in A/\mathfrak{m}^{n+2}$ such that $a_n = b \mod \mathfrak{m}^{n+1}$. This implies in particular that $0 = f_n(a_n) = f_{n+1}(b) \mod \mathfrak{m}^{n+1}$, so that $f_{n+1}(b)$ is in $\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Then $a_0 = b \mod \mathfrak{m}$.

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
b & \in & A/\mathfrak{m}^{n+2} \\
\downarrow & & \downarrow \\
a_n & \in & A/\mathfrak{m}^{n+1} \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
a_0 & \in & A/\mathfrak{m}
\end{array}
$$

Next, we (try to) define

$$a_{n+1} := b - \frac{f_{n+1}(b)}{f'_{n+1}(b)}.$$

Note that $f'_{n+1}(b)$ is a invertible in $A/\mathfrak{m}^{n+2}$, since a_0 being a simple root of f_0 means $0 \neq f'_0(a_0) = f'_{n+1}(b) \mod \mathfrak{m}$ is a unit in k, meaning $f'_{n+1}(b)$ has a representative not in $\mathfrak{m}$, whence it is represented by a unit in A, and is therefore invertible. Compute[1]

$$f_{n+1}(a_{n+1}) = f_{n+1}(b) - f_{n+1}(b) = 0$$

in $A/\mathfrak{m}^{n+2}$, as well as $a_n = a_{n+1} \mod \mathfrak{m}^{n+1}$. Thus $\{a_n\}_{n\geq 0}$ is a compatible system of elements giving rise to an element a of $\lim_n A/\mathfrak{m}^n = A$, with $f(a) = 0$, and $\bar{a} = a_0$, and we are done. □

We won't use the following lemma (at least not in these notes), but we include it as it is interesting and fits nicely into the themes explored.

Lemma 4.5. *Let $(A, \mathfrak{m}, k)$ be a henselian local ring. Then the category* $\mathrm{FinEt}(A)$ *of finite étale algebras over A is equivalent to* $\mathrm{FinEt}(k)$ *via the functor* $(A \to B) \mapsto (k \to B/\mathfrak{m}B)$.

Proof. See [stacks, Tag 04GK]. □

4.2 Henselisation and stalks

The third characterisation of henselian rings (3. of Prop. 4.3) gives us an idea of how to construct a henselian ring out of any local ring. Roughly, we just take the colimit over the étale algebras described there. We will drop a lot of details in this subsection, but they may be picked up again at [stacks, Tag 0BSK].

Lemma 4.6 (Henselisation)**.** *If $(A, \mathfrak{m}, k)$ is a local ring, then there exists a local morphism $A \to A^h$ such that*

1. *A^h is henselian,*
2. *A^h is a filtered colimit of étale A-algebras,*

[1] I have not done this computation and so am trusting the stacks project about it for now!

3. $\mathfrak{m}A^h$ *is maximal,*

4. $k = A^h/\mathfrak{m}A^h$.

Remark. Such an A^{h} is unique up to unique isomorphism. It is the *henselisation* of A.

Construction (sketch). The filtered category we will take a colimit over has objects $(B, \mathfrak{q})$ where $A \to B$ is an étale algebra, $\mathfrak{q} \subset B$ lies over $\mathfrak{m}$, and $k = \kappa(\mathfrak{q})$. Morphisms $(B, \mathfrak{q}) \to (B', \mathfrak{q}')$ are A-algebra morphisms $\psi\colon B \to B'$ such that $\psi^{-1}(\mathfrak{q}') = \mathfrak{q}$.

- This is a filtered category. We set $A^{\mathrm{h}} := \operatorname{colim}_{(B,\mathfrak{q})} B$.
- Elements $x \in A^{\mathrm{h}}$ are represented by triples $(B, \mathfrak{q}, f)$ where $f \in B$. Two representatives $(B, \mathfrak{q}, f)$ and $(B', \mathfrak{q}', f')$ give the same element x if they eventually become equal, meaning if there exists $(B'', \mathfrak{q}'')$, hit by morphisms ψ and ψ' from $(B, \mathfrak{q})$ and $(B', \mathfrak{q}')$ respectively, such that $\psi(f) = \psi'(f')$. (Cf. germes of a sheaf.)
- We may choose representatives such that $\mathfrak{m}B = \mathfrak{q}$, and $\mathfrak{q}$ is the only prime in B over $\mathfrak{m}$.
- An element x not in the ideal $\mathfrak{m}A^{\mathrm{h}}$ is represented by $(B, \mathfrak{q}, f)$ such that $f \notin \mathfrak{m}B = \mathfrak{q}$. Such an x has an inverse given by $(B_{\mathfrak{q}}, \mathfrak{q}S_{\mathfrak{q}}, 1/f)$. In particular, it is a unit. Thus:
- $\mathfrak{m}A^{\mathrm{h}}$ is the unique maximal ideal of A^{h}.
- A^{h} is henselian (proof omitted).

□

Given a separable closure k^{sep} of k, we can modify the colimit above slightly to produce a strictly henselian ring.

Lemma 4.7 (Strict henselisation)**.** *Given $(A, \mathfrak{m}, k)$ local and $k \subseteq k^{sep}$ a separable closure, there exists a commutative diagram*

$$\begin{array}{ccccc} k & = & k & \longrightarrow & k^{sep} \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A^h & \longrightarrow & A^{sh} \end{array}$$

where the left square is from the previous lemma, and

1. $A^h \to A^{sh}$ *is a local morphism,*

2. A^{sh} *is henselian,*

3. A^{sh} *is a filtered colimit of étale A-algebras,*

4. $\mathfrak{m}A^{sh}$ *is maximal,*

5. $k^{sep} = A^{sh}/\mathfrak{m}A^{sh}$ *(in particular, A^{sh} is strictly henselian).*

Remark. The *strict henselisation* A^{sh} of A, with respect to k^{sep}/k, is unique up to unique isomorphism.

Construction (sketch). We do the same thing as for henselisation with the following modification. Instead of pairs $(B, \mathfrak{q})$, we take triples $(B, \mathfrak{q}, \varphi)$ such that $\varphi\colon \kappa(\mathfrak{q}) \to k^{\mathrm{sep}}$ is a k-algebra morphism. □

This colimit should remind us of the description of $\mathcal{O}_{Y,y}$ that we produced in the interlude.

Lemma 4.8. *Let A be a ring and $\mathfrak{p} \subset A$ prime. Let $\kappa(\mathfrak{p}) \subseteq \kappa^{sep}$ be a separable closure. We have a filtered category of triples $(B, \mathfrak{q}, \varphi)$ where $A \to B$ is étale, $\mathfrak{q}$ is a prime of B over $\mathfrak{p}$, and $\varphi \colon \kappa(\mathfrak{q}) \to \kappa^{sep}$ is a $\kappa(\mathfrak{p})$-algebra morphism, and canonical isomorphisms*

$$(A_{\mathfrak{p}})^{sh} = \operatorname*{colim}_{(B,\mathfrak{q},\varphi)} = \operatorname*{colim}_{(B,\mathfrak{q},\varphi)} S_{\mathfrak{q}}.$$

Proof. See [stacks, Tag 04GW]. □

Lemma 4.9. *The stalk $\mathcal{O}_{Y,\bar{y}}$ is the strict henselisation $\mathcal{O}_{Y,y}^{sh}$ of the Zariski stalk $\mathcal{O}_{Y,y}$ at the point $y \in Y$ under $\bar{y}$.*

Proof. Recall the colimit

$$\mathcal{O}_{Y,\bar{y}} = \mathcal{O}_{\operatorname{Spec} A,\bar{y}} = \operatorname*{colim}_{(B,\mathfrak{q},\varphi)} B,$$

indexed by $(B, \mathfrak{q}, \varphi)$ such that $A \to B$ is an étale algebra, $\mathfrak{q} \subset B$ a prime over $\mathfrak{p}$ (the prime under $\bar{y}$), and $\varphi \colon \kappa(\mathfrak{q}) \to k$ is a $\kappa(\mathfrak{p})$-algebra morphism. Note that the embedding $\kappa(\mathfrak{p}) \subseteq k$ gives us a separable closure $\kappa(\mathfrak{p}) \subseteq \kappa^{\mathrm{sep}} \subseteq k$. We get that $\varphi \colon \kappa(\mathfrak{q}) \to k$ factors through $\kappa(\mathfrak{q} \to \kappa^{\mathrm{sep}}$. We may thus apply Lemma 4.8 and conclude the result. □

The following technical lemma is good to know and might be useful in later talks.

Lemma 4.10 (Tag 06LJ)**.** *If A is noetherian then so are A^h and A^{sh}.*

5 Finishing the proof

Theorem 5.1 (Tag 03QO)**.** *Let $(A, \mathfrak{m}, k)$ be a strictly henselian local ring. Then, for any sheaf $\mathscr{F}$ on $(\operatorname{Spec} A)_{ét}$, the global sections and the stalk at $\mathfrak{m}$ coincide,*

$$\Gamma(\operatorname{Spec} A, \mathscr{F}) = \mathscr{F}_{\mathfrak{m}}.$$

Proof. Let $(U = \operatorname{Spec} R, \bar{u})$ be an affine étale neighbourhood of $\mathfrak{m}$. Then $\operatorname{Spec} R \to \operatorname{Spec} A$ is étale (i.e. $A \to R$ is étale), and $\kappa(\bar{u}) = \kappa(\mathfrak{m}) = k$ (as k is separably closed). Thus, by Prop. 4.3, we have a section $\tau \colon A \to R$ with $\tau^{-1}(\mathfrak{m}) = \mathfrak{q}$, where $\mathfrak{q}$ is the point under $\bar{u}$. Thus, the étale neighbourhood $(\operatorname{Spec} A, \mathfrak{m}) \to (\operatorname{Spec} A, \mathfrak{m})$ is cofinal in $\mathscr{N}_{\mathfrak{m}}^{\mathrm{aff}}$, and we are done. □

Corollary. *The global sections functor $\Gamma(\operatorname{Spec} A, -) : \mathrm{Ab}(\operatorname{Spec} A)_{ét} \to \mathrm{Ab}$ is exact.*

Proof. The functor $\mathscr{F} \mapsto \mathscr{F}_{\mathfrak{m}}$ is exact. □

Corollary. *If $A \to B$ is finite then $H^q_{ét}(\operatorname{Spec} B, \mathscr{G}) = 0$ for all $q \geq 1$, for all abelian sheaves $\mathscr{G}$ on $\operatorname{Spec} B$.*

Proof. By Lemma 4.2, B is a finite product of strictly henselian local rings $(B_i, \mathfrak{m}_i)$. Thus,

$$H^q_{\text{ét}}(\operatorname{Spec} B, \mathscr{G}) = H^q_{\text{ét}}(\coprod_i \operatorname{Spec} B_i, \mathscr{G}) = \prod_i H^q_{\text{ét}}(\operatorname{Spec} B_i, \mathscr{G}).$$

Now by the previous corollary, each piece $H^q_{\text{ét}}(\operatorname{Spec} B_i, \mathscr{G})$ is zero, so we are done. □

Proof of the Key Lemma 2.2. By Lemma 4.9, $\mathcal{O}_{Y,\bar{y}}$ is strictly henselian. Apply the last corollary. □

References

[AM69] Michael F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* English. Reading, Mass.-Menlo Park, Calif.-London-Don Mills, Ont.: Addison-Wesley Publishing Company (1969). 1969.

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