Étale morphisms

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We mostly follow Bhatt's notes [1].

Definition. A local homomorphism of local rings $f : (B, \mathfrak{n}) \to (A, \mathfrak{m})$ is called **unramified** if $f(\mathfrak{n})B = \mathfrak{m}$ and $\kappa(\mathfrak{m})$ is a finite separable extension of $\kappa(\mathfrak{n})$.

Definition. A morphism of schemes $\pi : X \to Y$ is called **unramified** at $x \in X$ if

(i) the local homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is unramified,

(ii) π is locally of finite type at x.

If π is unramified at all $x \in X$, it is called **unramified**.

Lemma 1. *Suppose A is a finitely generated algebra over a field k with $\Omega_{A/k} = 0$. Then A is a finite direct sum of finite separable field extensions of k.*

Sketch of proof. First assume $k = \bar{k}$. Then for any prime $\mathfrak{p} \subset A$ and any maximal ideal $\mathfrak{m} \subset A$ containing $\mathfrak{p}$,

$$\mathfrak{m}_\mathfrak{m} / \mathfrak{m}_\mathfrak{m}^2 \cong k \otimes_{A_\mathfrak{m}} \Omega_{A_\mathfrak{m}/k} = 0.$$

By Nakayama's lemma, it follows that $\mathfrak{m}_\mathfrak{m} = \mathfrak{p}_\mathfrak{m} = 0$. Varying $\mathfrak{p}$ and $\mathfrak{m}$, we deduce that A is a reduced artinian k-algebra, hence a finite direct sum of copies of k.
Deduce the case of arbitrary k using a base change

$$\begin{array}{ccc} A \otimes_k \bar{k} & \longleftarrow & A \\ \uparrow & \square & \uparrow \\ \bar{k} & \longleftarrow & k. \end{array}$$

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Theorem 2. *Suppose $\pi : X \to Y$ is locally of finite type. Then for any $x \in X$, the following are equivalent:*

(i) π is unramified at x.

(ii) $\Omega_{X/Y,x} = 0$.

(iii) There exists an open $x \in U$ and a locally closed embedding $j : U \hookrightarrow \mathbb{A}^n_Y$ defined by an ideal sheaf $\mathcal{I}$ such that $\Omega_{\mathbb{A}^n_Y/Y,x}$ is generated by dg for sections g of $\mathcal{I}$.

(iv) There exists an open $x \in U$ such that $\operatorname{diag}_{X/Y}\big|_U$ is an open embedding.

Sketch of proof. (i) $\implies$ (ii). Consider a homomorphism $B \to A$ and primes $\mathfrak{p} \in \operatorname{Spec} A$, $\mathfrak{q} = \mathfrak{p} \cap B$. If $B_\mathfrak{q} \to A_\mathfrak{p}$ is an unramified homomorphism of local rings, we have a cartesian diagram

$$\begin{array}{ccc} \kappa(\mathfrak{p}) & \longleftarrow & A_\mathfrak{p} \\ \uparrow & \square & \uparrow \\ \kappa(\mathfrak{q}) & \longleftarrow & B_\mathfrak{q}. \end{array}$$

It follows that

$$\Omega_{A/B,\mathfrak{p}} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}) = \Omega_{A_\mathfrak{p}/B_\mathfrak{q}} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}) = \Omega_{\kappa(\mathfrak{p})/\kappa(\mathfrak{q})} = 0.$$

(ii) $\implies$ (i). Use Lemma 1.
(ii) $\iff$ (iii). Use the conormal exact sequence

$$j^*(\mathcal{I}/\mathcal{I}^2) \to j^*\Omega_{\mathbb{A}^n_Y/Y} \to \Omega_{U/Y} \to 0.$$

(ii) $\iff$ (iv). We show that for any affine opens $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset \pi^{-1}(\operatorname{Spec} B)$, the closed embedding $\operatorname{Spec} A \to \operatorname{Spec} A \otimes_B A$ is actually an open embedding if and only if $\Omega_{A/B} = 0$. To this end, apply the following lemma to the ideal $\ker(A \otimes_B A \to A)$.

Lemma 3. *Suppose R is a ring and $I \subset R$ is a finitely generated ideal. If $I^2 = I$, then $V(I) = D(e)$ for an idempotent element $e \in R$.*

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Proposition 4. *Unramified morphisms are stable under base change and composition. A morphism that is locally of finite type is unramified if and only if all its fibers are unramified.*

Sketch of proof. Use Theorem 2.(ii). ■

Proposition 5. *(i) If for morphisms $f : X \to Y$ and $g : Y \to Z$ the composition gf is unramified, then so is f.*

(ii) Every monomorphism locally of finite type is unramified.

Sketch of proof. (i) Use Theorem 2.(ii).

(ii) Use Theorem 2.(iv).

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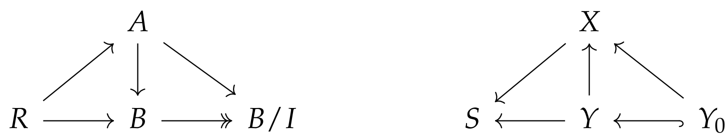
Theorem 6. *Suppose $\pi : X \to S$ is locally of finite type. Then π is unramified if and only if for every affine morphism $Y \to S$ and every closed subscheme $Y_0 \subset Y$ defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$, the map*

$$\operatorname{Mor}_S(Y, X) \to \operatorname{Mor}_S(Y_0, X)$$

is injective.

Sketch of proof. Reduce to the affine case as in the following diagrams:



Fix a homomorphism $A \to B/I$. The trick is to notice that *differences* of factorizations $A \to B$ correspond to derivations $A \to I$.
For the backward implication, consider $B := (A \otimes_R A)/J^2$, where $J = \ker(A \otimes_R A \to A)$, as well as the ideal $I := J/J^2$. ∎

Definition. A morphism of schemes $\pi : X \to Y$ is called **étale** at $x \in X$ if π is unramified and flat at x.
It is called **étale** if it is étale at all points $x \in X$.

Proposition 7. *Consider morphisms $f : X \to Y$ and $g : Y \to Z$. If g is unramified and gf is étale, then f is étale.*

Sketch of proof. Use Proposition 5 and the fiberwise criterion for flatness, Theorem 17 of Sebastian's talk. ∎

Theorem 8. *A morphism $\pi : X \to Y$ is étale if and only if the following holds:*

(i) There exists an open $x \in U$ and a locally closed embedding $j : U \hookrightarrow \mathbb{A}^n_Y$.

(ii) If $\mathcal{I}$ is the corresponding ideal sheaf, then there exist sections $g_1, \ldots, g_n$ of $\mathcal{I}$ such that the $dg_1, \ldots, dg_n$ form a basis for $\Omega_{\mathbb{A}^n_Y/Y,x} \otimes_{\mathcal{O}_{\mathbb{A}^n_Y,x}} \kappa(x)$.

Sketch of proof. (ii) $\Longrightarrow$ (i). Unramifiedness follows from Theorem 2. Flatness uses the theory of Cohen-Macaulay rings. See for example the exposition in [3, Section 25.2.1].
(i) $\Longrightarrow$ (ii). See for example [2, Tag 00UE].

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We record some more properties of étale morphisms that follow quickly from the properties of unramified and flat morphisms:

Proposition 9. *Étale morphisms are open.*

Sketch of proof. In fact, flat morphisms locally of finite type are open (Theorem 7 of Sebastian's talk). ∎

Proposition 10. *Étale morphisms are stable under base change and composition.*

Proposition 11. *Étale morphisms are quasi-finite.*

Proposition 12. *A flat morphism locally of finite type is étale if and only if it is unramified.*

Finally, we present an analog of Theorem 6 for étale morphisms:

Theorem 13. *Suppose $\pi : X \to S$ is locally of finite type and separated. Then π is étale if and only if for every affine morphism $Y \to S$ and every closed subscheme $Y_0 \subset Y$ defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$, the map*

$$\mathrm{Mor}_S(Y, X) \to \mathrm{Mor}_S(Y_0, X)$$

is bijective.

Sketch of proof. <u>(i) $\Longrightarrow$ (ii)</u>. Reduce to the case $Y = S$ via base change. Let $\varphi : X \to Y$ be the structure morphism. Then the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;s\;} & X \\ {\scriptstyle s}\downarrow & \square & \downarrow{\scriptstyle (s\varphi, id)} \\ X & \xrightarrow[\mathrm{diag}_{X/Y}]{} & X \times_Y X \end{array}$$

shows that every section $s : Y \to X$ is an isomorphism onto a connected component of X. Now consider a morphism $t \in \mathrm{Mor}_Y(Y_0, X)$. Since the underlying sets of Y_0 and Y are the same, there is a connected component X_i of Y such that $X_i \to Y$ is a universal homeomorphism. Also, $X_i \to Y$ is étale. Now the faithfully flat base change

$$\begin{array}{ccc} X_i \times_Y X_i & \longrightarrow & X_i \\ \downarrow & \square & \downarrow \\ X_i & \longrightarrow & Y. \end{array}$$

shows that $X_i \to Y$ is in fact an isomorphism. The inverse is our desired extension of t to Y.
<u>(ii) $\Longrightarrow$ (i)</u>. Assume $S = \mathrm{Spec}\, R$, $X = \mathrm{Spec}\, R[\underline{X}]/I$. Using the hypothesis, find a splitting of the exact sequence

$$0 \to I/I^2 \to R[\underline{X}]/I^2 \to R[\underline{X}]/I \to 0.$$

The resulting map $R[\underline{X}]/I^2 \to I/I^2$ is a derivation, so induces a map $\Omega_{R[\underline{X}]/R} \otimes_{R[\underline{X}]} R[\underline{X}]/I \to I/I^2$ that is inverse to $I/I^2 \to \Omega_{R[\underline{X}]/R} \otimes_{R[\underline{X}]} R[\underline{X}]/I$. ■

References

[1] Bhatt, B., *The étale topology*. `http://www-personal.umich.edu/~bhattb/math/etalestcksproj.pdf`, 2018.

[2] Stacks Project authors, *The Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[3] Vakil, R., *The Rising Sea: Foundations of Algebraic Geometry*. Preprint, 2017.