Seminar Étale Cohomology
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Derived Functors

1. Additive categories

Def. A *pre-additive category* is a category together with the structure of an abelian group on each Hom set such that composition is bilinear.

Prop.-Def. In any pre-additive category, an object is initial if and only if it is final. Such an object is called a *null object*.

Prop.-Def. In any pre-additive category, an object is a product of two objects X and Y if and only if it is their coproduct (with appropriate morphisms). Such an object is called a *biproduct* or *direct sum* $X \oplus Y$.

Def. An *additive category* is a pre-additive category with a null object and all direct sums.

Equivalent: It contains all finite direct sums, including the empty one.

2. Abelian categories

Consider an additive category $\mathcal{C}$ and a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Def. monomorphism, kernel $\ker(f)$

Fact. Every kernel is a monomorphism.

Def. epimorphism, cokernel $\mathrm{coker}(f)$

Fact. Every cokernel is an epimorphism.

Def. image $\mathrm{im}(f) := \ker(Y \to \mathrm{coker}(f))$.

Def. coimage $\mathrm{coim}(f) := \mathrm{coker}(\ker(f) \to X)$.

Def. natural morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$.

Def. An *abelian category* is an additive category with all kernels and cokernels and for which all the above morphisms $\mathrm{coim}(f) \to \mathrm{im}(f)$ are isomorphisms.

Note. The last condition is equivalent to requiring that every monomorphism is a kernel and that every epimorphism is a cokernel.

Note. All the usual diagram lemmas hold in any abelian category.

3. Examples

The category **Ab** of abelian groups.

The category $\mathbf{Mod}_R$ of left modules over a ring R.

The category of sheaves of abelian groups on a topological space.

The category $\mathbf{Mod}_{\mathcal{O}_X}$ of sheaves of modules on a locally ringed space $(X, \mathcal{O}_X)$.

The category $\mathbf{QCoh}_{\mathcal{O}_X}$ of quasi-coherent $\mathcal{O}_X$-modules on a scheme X.

The *diagram category* of functors $X \to \mathcal{C}$ for a small category X and an abelian category $\mathcal{C}$.

The category of all chain complexes in an abelian category $\mathcal{C}$.

The opposite category $\mathcal{C}^{\text{opp}}$ of an abelian category $\mathcal{C}$.

Note. Passing to the opposite category interchanges kernels with cokernels, images with coimages, projectives with injectives, and so on.

For the following we fix an abelian category $\mathcal{C}$.

4. Projectives and injectives

Fact. For any object X the functor $\mathcal{C} \to \mathbf{Ab}$, $Y \mapsto \operatorname{Hom}_{\mathcal{C}}(X, Y)$ is left exact.

Def. An object X is *projective* if and only this functor is exact.

Prop. Every free module is projective in $\mathbf{Mod}_R$.

Caution. In general there is no good notion of a free object in $\mathcal{C}$.

Def. *enough projectives.*

Prop. The category $\mathbf{Mod}_R$ has enough projectives.

Fact. For any object X the functor $\mathcal{C} \to \mathbf{Ab}$, $Y \mapsto \operatorname{Hom}_{\mathcal{C}}(Y, X)$ is left exact.

Def. An object X is *injective* if and only this functor is exact.

Prop. An abelian group is injective if and only if it is divisible.

Ex. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}[\frac{1}{n}]/\mathbb{Z}$ are injective in **Ab**.

Def. *enough injectives.*

Prop. The category **Ab** has enough injectives.

Prop. The category $\mathbf{Mod}_R$ has enough injectives.

Prop. The category $\mathbf{Mod}_{\mathcal{O}_X}$ has enough injectives.

5. Resolutions

Def. *Resolution (to the right)* $0 \to X \to Y^0 \to Y^1 \to Y^2 \to \ldots$ or in short $0 \to X \to Y^\bullet$.

Def. A resolution is called <*adjective*> if and only if each Y^n is <adjective>.

Prop. If $\mathcal{C}$ has enough injectives, every object possesses an injective resolution.

Prop. Consider any resolution $0 \to X \to Z^\bullet$ and any injective resolution $0 \to Y \to J^\bullet$.

(a) Any morphism $f : X \to Y$ extends to a morphism of complexes $(X \to Z^\bullet) \to (Y \to J^\bullet)$.

(b) Any two such extensions $Z^\bullet \to J^\bullet$ are equivalent under a homotopy.

6. δ-Functors

Def. *δ-functor $T^\bullet$*

Def. *morphism* of δ-functors

Def. *universal δ-functor*

Prop. Any universal δ-functor $T^\bullet$ is determined up to unique isomorphism by T^0.

7. Derived functors

Now assume that $\mathcal{C}$ has enough injectives, and consider a left exact covariant additive functor $F \colon \mathcal{C} \to \mathcal{D}$ to another abelian category $\mathcal{D}$.

Construction. For any object X choose an injective resolution $0 \to X \to I_X^\bullet$. For any integer $i \geqslant 0$ set $R^iF(X) := H^i(F(I_X^\bullet))$. For any morphism $f : X \to Y$ choose an extension $I_f^\bullet$ to a morphism of complexes $(X \to I_X^\bullet) \to (Y \to I_Y^\bullet)$. For any integer $i \geqslant 0$ set $R^iF(f) := H^i(F(I_f^\bullet)) : R^iF(X) \to R^iF(Y)$.

Thm.-Def. This is a universal δ-functor with $R^0F \cong F$, called the *(right) derived functor of F*.

Variant. For a contravariant left exact functor $F : \mathcal{C} \to \mathcal{D}$ one applies this to the covariant left exact functor $F : \mathcal{C}^{\text{opp}} \to \mathcal{D}$. Since injective right resolutions in $\mathcal{C}^{\text{opp}}$ correspond to projective left resolutions in $\mathcal{C}$, one must assume that $\mathcal{C}$ has enough projectives, and obtains the *(right) derived functor of F*, again denoted by R^iF.

Variant. For a covariant right exact functor $F : \mathcal{C} \to \mathcal{D}$ one applies this to the covariant left exact functor $F : \mathcal{C}^{\text{opp}} \to \mathcal{D}^{\text{opp}}$. Again one works with projective left resolutions in $\mathcal{C}$, must assume that $\mathcal{C}$ has enough projectives, and obtains the *(left) derived functor of F*, which is now denoted by L_iF.

8. Acyclic resolutions

Consider any δ-functor $T^\bullet\colon \mathcal{C} \to \mathcal{D}$.

Def. An object X of $\mathcal{C}$ is called $T^\bullet$*-acyclic*, or just T^0*-acyclic* if $T^\bullet$ is the derived functor of T^0, if $T^i(X) = 0$ for all $i > 0$.

Note. If $\mathcal{C}$ has enough injectives and $T^\bullet$ is a derived functor, every injective is $T^\bullet$-acyclic. But many δ-functors $T^\bullet$ possess more acyclic objects, and then we can compute them using acyclic objects instead of injective ones.

Prop. For any object X and any $T^\bullet$-acyclic resolution $0 \to X \to A^\bullet$ in $\mathcal{C}$, for every $i \geqslant 0$ there is a natural isomorphism $T^i(X) \cong H^i(T^0(A^\bullet))$.

9. Flabby sheaves

Now consider a scheme X. By §3 the category $\mathbf{Mod}_{\mathcal{O}_X}$ has enough injectives. Thus the left exact functor $\Gamma(X, \_)\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Ab}$ possesses the right derived functors

$$H^i(X, \_) := R^i\Gamma(X, \_).$$

Def. *flabby* sheaf of $\mathcal{O}_X$-modules.

Prop. For any short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we have:

(a) If $\mathcal{F}'$ is flabby, then for every open subset $U \subset X$ the sequence of sections over U is exact: $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$.

(b) If $\mathcal{F}'$ and $\mathcal{F}$ are flabby, then so is $\mathcal{F}''$.

Prop. Any injective $\mathcal{O}_X$-module is flabby.

Prop. Any flabby $\mathcal{O}_X$-module is acyclic for $H^i(X, \_)$.

Cor. For any flabby resolution $0 \to \mathcal{F} \to \mathcal{G}^\bullet$ in $\mathbf{Mod}_{\mathcal{O}_X}$, for every $i \geqslant 0$ there is a natural isomorphism $H^i(X, \mathcal{F}) \cong H^i(\Gamma(X, \mathcal{G}^\bullet))$.

Prop. For any flabby $\mathcal{O}_X$-module $\mathcal{F}$ and any morphism $f\colon X \to Y$ the $\mathcal{O}_Y$-module $f_*\mathcal{F}$ is flabby.

Now assume that $X = \operatorname{Spec} A$ for a noetherian ring A.

Prop. For any injective A-module I the $\mathcal{O}_X$-module $\tilde{I}$ is flabby.

Prop. Any quasicoherent sheaf is acyclic for $H^i(X, \_)$.

10. Čech cohomology

Consider a separated noetherian scheme X with a finite open affine covering $\mathcal{U} = (U_i)_{i\in I}$.

Construction. The *sheafified Čech complex* $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$ for any $\mathcal{O}_X$-module $\mathcal{F}$.

Prop. This yields a resolution $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$.

Note. $\Gamma(X,\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}))$ is just the usual Čech complex of $\mathcal{F}$ with respect to $\mathcal{U}$, and its cohomology $\check{H}^i(\mathcal{U},\mathcal{F}) := H^i(\Gamma(X,\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})))$ is the usual Čech cohomology. So we have two δ-functors

$$\begin{aligned} \check{H}^i(\mathcal{U},\_) :&\ \ \mathbf{Mod}_{\mathcal{O}_X} \longrightarrow \mathbf{Ab}, \\ H^i(X,\_) :&\ \ \mathbf{Mod}_{\mathcal{O}_X} \longrightarrow \mathbf{Ab}, \end{aligned}$$

which are isomorphic in degree $i = 0$.

Prop. Every quasicoherent sheaf on X can be embedded in a flabby quasicoherent sheaf.

Prop. Every flabby quasicoherent sheaf is acyclic for $\check{H}^i(\mathcal{U},\_)$.

Thm. For any quasicoherent sheaf $\mathcal{F}$ and every $i \geqslant 0$ there is a natural isomorphism

$$\check{H}^i(\mathcal{U},\mathcal{F}) \cong H^i(X,\mathcal{F}).$$

Note. In particular the restriction to the abelian subcategory $\mathbf{QCoh}_{\mathcal{O}_X}$ of the derived functor of $\Gamma(X,\_) : \mathbf{Mod}_{\mathcal{O}_X} \longrightarrow \mathbf{Ab}$ is isomorphic to the derived functor of the restriction $\Gamma(X,\_) : \mathbf{QCoh}_{\mathcal{O}_X} \longrightarrow \mathbf{Ab}$.